Consensus of Linear Multi-Agent Systems with Communication and Input Delays

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Abstract In this paper, we consider the consensus problem of a group of general linear agents with communication and input delays under a fixed, undirected network topology. By factorizing the characteristic equation and the multi-agent system into a set of reduced-order factors, the problem is transformed to the stability analysis of resulting factors with reduction in complexity. Furthermore, stable ranges of the control gain, such that the consensus of multi-agent systems could be reached when delays vanish, are analyzed. With control gain confined to stable ranges, and applying the advanced clustering with frequency sweeping method to investigate the stability of factors, the delay-independent and delay-dependent consensus are discussed. An illustrative example is offered to verify the analytical conclusions.

Key words Multi-agent systems, consensus, time delays, stability analysis


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Consensus problems of multi-agent systems have been paid substantial attentions in many recent investigations due to their important applications. The goal of the consensus problem is to make the networked system achieve collective behaviors such as maintaining a formation, swarming or reaching a consensus by appropriate control law for each agent. Recently, many studies have focused on consensus problems for the first-order (integrators) or second-order (double integrators) multi-agent systems, considering fixed and switching communication topologies[14]. Based on these researches, [2] studied the consensus of a special higher-order multi-agent system consisting of integrators by inspecting stable consensus regions. Reference [3] extended the results to a general higher-order multi-agent system by a Lyapunov-based method. And [4–5] alternatively designed some distributed observer-type control law for each agent with general linear dynamics.

Due to unavoidable time delays in networks, some researchers further considered the consensus behaviors of multi-agent systems under direct and undirected information flow with presence of delays. Works tackled the consensus problems of the first-order, second-order and general linear[16] multi-agent systems with communication or input delays were presented via time-domain (Lyapunov theorems) and frequency-domain (the Nyquist stability criterion) approaches. Reference [11] analyzed the case of second-order system with disturbances. Moreover, [12–13] performed deep discussions around the first and second order agents with delays and applied the cluster treatment of characteristic roots paradigm. A distinct approach based on the stability and bifurcation analysis was taken in [14] to handle a system with linearized general dynamics and time delay. For the stability analysis of consensus in the delay space, time-domain (Lyapunov theorems) methods are based on the existence of feasible solutions of linear matrix inequalities, from which it is difficult to obtain the time-delay boundaries, and only conservative bounds in time delay are yielded for stable operations. A recently reported method, called advanced clustering with frequency sweeping (ACSFS) method, is efficient and convenient for the stability analysis of time-delay systems in the entire delay-parameter space[15–16]. It provides an analytical approach which enables to test the delay-independent stability of system and can extract stability boundaries in two-delay domain easily.

In this paper, we discuss the consensus problem of a multi-agent system with general linear dynamics. The proposed neighbor-based control input for each agent considers both input and communication delays, which makes the closed loop system distinctive from that in existing literature. A key factorization of system characteristic equation reduces the consensus problem of multi-agent system to the stability analysis of a number of reduced-order factors, where the order of the system characteristic equation relies on the number of agents as well as the order of each agent, but the order of the factor is equal to that of each agent. Thus the complexity is dramatically alleviated. Moreover, we investigate the selection of control gain which makes factors stable when delays vanish. When control gain is confined to stable ranges, we further carry out the stability analysis factors in the delay parametric space. For the stability analysis of every factor, ACSFS method is employed to check the delay-independent/delay-dependent stability and to detect the stability boundaries in delay parametric space. The main contributions of this work are the factorization of system characteristic equation, and declarations of delay parametric space produced by ACSFS method, from which allowable delay variations could be determined for stable consensus of the multi-agent system.

The paper is organized as follows. In Section 1, the formulation of the problem is presented. Section 2 presents a brief introduction to ACSFS method. The stability analysis, including the factorization of the characteristic equation, the selection of control gain and deployment of ACSFS method, is presented in Section 3. Section 4 offers an illustrative example. The conclusions are given in Section 5.

Throughout the paper, the following notations are used. Let $\mathcal{R}, \mathbb{R}_0, \mathbb{N}$ denote the set of real numbers, the set of nonnegative real numbers and the set of natural numbers (including zero), respectively. Let $\mathbb{R}^{m \times n}$ be the set of $n \times n$ real matrices. The superscript $T$ means transpose for real matrices. $\| \cdot \|$ denotes the induced two-norm. $I_n$ is the $n$-dimensional identity matrix. Let $1_N \in \mathbb{R}^N$ denote the $N$-dimensional vector with all entries equal to one. $\otimes$ denotes the Kronecker product. For $s \in \mathbb{C}$, $\Re(s)$ and $\Im(s)$ represent its real part and imaginary part, respectively. Matrices are assumed to have compatible dimensions.

1 Problem formulation

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ be a weighted undirected network of order $N$, where $\mathcal{V}$ is the set of nodes $\mathcal{V} = \{v_1, v_2, \ldots, v_N\}$, $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the set of arcs and $\mathcal{A}$ is the weighted adjacency matrix $\mathcal{A}(\mathcal{G}) = [a_{ij}]$. An arc $e_{ij}$ in the network is denoted by an unordered pair of nodes $(v_i, v_j)$. By the definition of adjacency matrices for weighted graphs, weights $a_{ij} = a_{ji} > 0$ are all positive if and only if there is an arc $(v_i, v_j)$ in $\mathcal{G}$. In this paper, only positively weighted networks are...
considered and it is assumed that $a_{ii} = 0$ for $i = 1, \ldots, N$. An undirected graph $\mathcal{G}$ is connected if there exists a path between any pair of distinct nodes $v_i$ and $v_j$ $(i, j = 1, \ldots, N)$ in $\mathcal{G}$, i.e., there exists a sequence of arcs $(v_1, v_{p_1}), (v_{p_1}, v_{p_2}), \ldots, (v_{p_{q-1}}, v_{p_q}, v_j)$ in the network with distinct nodes $v_{p_q}, q = 1, 2, \ldots, l$. The out-degree of the node $v_i$ is defined as $d_i = \sum_{j=1}^{N} a_{ij}$ and we introduce $N \times N$ matrix $\Delta = \text{diag}(d_1, d_2, \ldots, d_N)$.

Consider a group of $N$ identical agents with general linear dynamics. The dynamics of the $i$-th agent are described as

$$\dot{x}_i(t) = Ax_i(t) + Bu_i(t), \quad t \geq 0$$

where $x_i(t) \in \mathbb{R}^n$ is the state vector of $i$-th agent, and $u_i(t), y_i(t) \in \mathbb{R}$ are the control input and the measured output, respectively. The topology among $N$ agents is represented by an undirected graph $\mathcal{G}$. The following assumption is used throughout the paper.

**Assumption 1.** The undirected graph $\mathcal{G}$ is connected.

**Definition 1.** The multi-agent system (1) is said to achieve consensus if for any well-defined initial conditions, there is a control input $u_i(t)$ such that the closed-loop system satisfies

$$\lim_{t \to \infty} ||x_i(t) - x_j(t)|| = 0, \quad i, j = 1, \ldots, N$$

where $\kappa > 0$ is the outer coupling strength which is to be determined, $d_i$ is the degree of agent $i$, $a_{ij}$ represents the topological structure of the network, and $\tau_{in}$ and $\tau_{com}$ are nonnegative input and communication delays, respectively. Let $\gamma_i = \tau_{in}$ and $\gamma_2 = \tau_{in} + \tau_{com}$. Under protocol (3), (1) can be rewritten as

$$\dot{z}_i(t) = Ax_i(t) - Bu_i(t) = Ax_i(t) - Bu_i(t) \sum_{j=1}^{N} a_{ij}[x_j(t - \tau_j) - x_i(t - \tau_2)]$$

where $\kappa > 0$ is the outer coupling strength which is to be determined, $d_i$ is the degree of agent $i$, $a_{ij}$ represents the topological structure of the network, and $\tau_{in}$ and $\tau_{com}$ are nonnegative input and communication delays, respectively. Let $\gamma_i = \tau_{in}$ and $\gamma_2 = \tau_{in} + \tau_{com}$. Under protocol (3), (1) can be rewritten as

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where $\kappa > 0$ is the outer coupling strength which is to be determined, $d_i$ is the degree of agent $i$, $a_{ij}$ represents the topological structure of the network, and $\tau_{in}$ and $\tau_{com}$ are nonnegative input and communication delays, respectively.

**Lemma 1** \textsuperscript{[17]} All eigenvalues of $G$ are real and have modulus smaller than or equal to 1. \textsuperscript{[14]} Matrix $G$ has a simple eigenvalue equal to 1 with $1_N$ as its corresponding eigenvector. Furthermore, there exists a vector $\gamma = [\gamma_1, \ldots, \gamma_N]^T \in \mathbb{R}^N$ such that

$$\gamma^T 1_N = 1, \quad \gamma^T (G - I_N) = 0$$

We denote the eigenvalues of $G$ as $\lambda_i, i = 1, \ldots, N$ and let $\lambda_1 = 1$.

In the sequel, some definitions and crucial conclusions related to ACFS method will be presented. Consider the characteristic function of a time delay system with two delays given by

$$p(s; \tau_1, \tau_2) = \sum_{k=0}^{J} P_k(s) e^{-s(\phi_k + \phi_{k+2}\tau_2)}$$

where $P_k(s)$ are polynomials in terms of $s$ with real coefficients, $\tau_1, \tau_2$ are constant nonnegative delays and $\phi_1, \phi_2 \in \mathbb{N}$. Assume that the largest power of $s$ is in $P_0(s)$ and $P_0(s)$ does not multiply any exponential functions, that is, $\phi_1 = \phi_2 = 0$. Therefore (6) is a retarded quasipolynomial. It is known that the stability of (6) may change only when $s = j\omega, \omega \in \mathbb{R}_0^+, \omega^2 = -1$, is a zero of (6). All nonnegative $\omega$ values, where $s = j\omega$ is a zero of (6) for some nonnegative delays, define the crossing frequency set

$$\Omega = \{ \omega \in \mathbb{R}_0^+ | p(j\omega; \tau_1, \tau_2) = 0, \text{ for some } \tau_1, \tau_2 \geq 0 \}$$

By introducing the Rekasius transformation,

$$\begin{align*}
\omega &= \frac{1}{sT_1} - \frac{1}{T_1 + sT_1}, \quad s = j\omega, \quad T_1 \in \mathbb{R}, \quad l = 1, 2
\end{align*}$$

the infinite-dimensional function (6) is converted to a finite-dimensional function with continuous coefficient. Upon substitution of (8) into (6) and with a multiply operation to remove the fractions, (6) is transformed as

$$g(j\omega; T_1, T_2) = (p(j\omega; \tau_1, \tau_2))^2 \prod_{l=1}^{2}(1 + j\omega T_l)$$

which is a function in terms of variables $\omega, T_1$ and $T_2$. Define the crossing frequency set of (9) as

$$\Omega = \{ \omega \in \mathbb{R}_0^+ | g(j\omega; T_1, T_2) = 0, \text{ for some } T_1, T_2 \in \mathbb{R} \}$$

**Lemma 2** \textsuperscript{[15–16]} The identity $\Omega \equiv \Omega$ holds.

**Remark 1.** Lemma 3 indicates that we can obtain $\Omega$ by finding $\Omega$ through the algebraic equation (9). Once $\Omega$ and the corresponding $T_1$ and $T_2$ are found, $\tau_1$ and $\tau_2$ can be obtained by the inverse transformation of (8) as

$$\begin{align*}
(\tau_1, \tau_2) &= \left(\frac{2\tan^{-1}(\omega T_1)}{\omega}, \frac{2\tan^{-1}(\omega T_2)}{\omega}\right) + (k_1, k_2) \frac{2\pi}{\omega}, \quad \omega \in \Omega, \quad k_1, k_2 \in \mathbb{N}
\end{align*}$$

**Definition 2.** Consider the two multivariate polynomials in terms of $\omega, \mu$, where $\mu = \{\mu_1, \mu_2\}$, with real coefficients

$$p_1 = \sum_{i=0}^{r_1} \phi_i(\omega, \mu_1) \mu_2^i = 0, \quad \phi_{r_1} \neq 0$$

$$p_2 = \sum_{i=0}^{r_2} \psi_i(\omega, \mu_1) \mu_2^i = 0, \quad \psi_{r_2} \neq 0$$
where $p_1$ and $p_2$ have positive degrees in terms of $p_2$ and $r_1, r_2 > 0$. Define $R_{p_2}(p_1, p_2)$, which is the resultant of $p_1$ and $p_2$ with respect to $\omega$ and $\mu_1$ by

$$R_{p_2}(p_1, p_2) = \frac{\phi_1}{\phi_0}, \frac{\phi_{r_1-1}}{\phi_0}, \ldots, \frac{\phi_{r_1}}{\phi_0}$$

that is the determinant of Sylvester matrix. The above $R_{p_2}(p_1, p_2)$ is of order $r_1 + r_2$; it has $r_2$ rows involving the $\phi$’s and $r_1$ rows involving the $\psi$’s.

Decompose (9) into the real part and imaginary part as

$$g(\omega, T_1, T_2) = g_{re}(\omega, T_1, T_2) + ig_{im}(\omega, T_1, T_2)$$

where $g_{re} = Re(g)$ and $g_{im} = Im(g)$. For $\omega$ to be a zero of (13), $g_{re}$ and $g_{im}$ should be simultaneously zero for some $(T_1, T_2)$.

$$g_{re} = \sum_{j=0}^{m} \alpha_j(\omega, T_1)T_2^j = 0$$

$$g_{im} = \sum_{j=0}^{m} \beta_j(\omega, T_1)T_2^j = 0$$

where $\alpha_j$ and $\beta_j$ are real polynomials in $\omega$ and $T_1$, and are assumed to have no common factors. And $m$ is the largest power of $T_2$ in $g_{re}$ and $g_{im}$. According to (12) define $R_{T_2}(g_{re}, g_{im})$, which is the resultant of $g_{re}$ and $g_{im}$ with respect to $\omega$ and $T_1$.

**Lemma 4:** If $(\omega, T_1, T_2)$ is a common zero of (14) and (15), then $R_{T_2}(g_{re}, g_{im}) = 0$. Conversely, if $R_{T_2}(g_{re}, g_{im}) = 0$, then at least one of the four conditions holds: 1) $\alpha_0(\omega, T_1) = \cdots = \alpha_m(\omega, T_1) = 0$; 2) $\beta_0(\omega, T_1) = \cdots = \beta_m(\omega, T_1) = 0$; 3) $\alpha_0(\omega, T_1) = \beta_0(\omega, T_1) = 0$; 4) there exists $(\omega, T_1, T_2)$ that is a common zero of (14) and (15).

**Remark 2.** Detection of the common zeros of (14) and (15) corresponds to the case 4) in Lemma 4 and the remaining three cases can be identified for a given $(\omega, T_1, T_2)$, which will be elaborated later. Therefore, we choose to investigate the zeros of $R_{T_2}(g_{re}, g_{im})$ instead of studying the zeros of $g(\omega, T_1, T_2)$.

If $R_{T_2}(g_{re}, g_{im}) = \sum_{j=0}^{q} \alpha_j(\omega)T_2^j$ and $\frac{\partial R_{T_2}}{\partial T_2} = \sum_{j=0}^{q} \beta_j(\omega)T_2^j$, where $q$ is the largest power of $T_2$ in $R_{T_2}$ and $\beta_j$, then according to (12) define $R(\omega, \frac{\partial R_{T_2}}{\partial T_2})$, which is the resultant of $R_{T_2}$ and $\frac{\partial R_{T_2}}{\partial T_2}$ with respect to $\omega$.

The following lemma presents the precise nonzero lower bound $\omega$ and upper bound $\bar{\omega}$ of $\Omega$ in the case of $\omega \neq 0$.

**Lemma 5:** The minimum and the maximum positive real roots of $R_{T_2}(\frac{\partial R_{T_2}}{\partial T_2}) = 0$, which can give rise to $(T_1, T_2) \in \mathbb{R}^2$ solutions in (13), are the exact lower and upper bounds of $\Omega$.

### 3 Main results

In this section, main results of consensus problem of (5) will be presented. For $x(t)$ in (5), introduce the following variable:

$$\delta(t) = x(t) - (I_N \gamma^T \otimes I_n)x(t)$$

where $\delta(t) = [\delta_1^T(t), \ldots, \delta_N^T(t)]^T$ and $\delta(t) \in \mathbb{R}^{Nn}$ satisfies $(\gamma^T \otimes I_n)\delta(t) = 0$, $\delta(t)$ is referred to as the disagreement vector and can be described by the following disagreement dynamics:

$$\dot{\delta}(t) = (I_N \otimes A)\delta(t) - (I_N \otimes B_{\text{kin}})\delta(t - \tau_1) + (G \otimes B_{\text{kin}})\delta(t - \tau_2)$$

First we can show that the consensus problem of (5) is equivalent to the asymptotical stability of the disagreement (17). Rewrite $\delta(t)$ as

$$\delta(t) = [(I_N - 1_N\gamma^T) \otimes I_n]x(t)$$

It can be easily found that 0 is a simple eigenvalue of $I_N - 1_N\gamma^T$ with 1 as its corresponding right eigenvector and 1 is another eigenvalue with multiplicity $N - 1$. It follows from (18) that $\delta(t) = 0$ if and only if $x_1(t) = x_2(t) = \cdots = x_N(t)$. That is to say, as $t \to \infty$ if $\delta(t) \to 0$ then the consensus problem of (5) is solved.

According to Lemmas 1 and 2, let $U_1 \in \mathbb{R}^{N \times (N-1)}$, $U_2 \in \mathbb{R}^{N(N-1) \times N}$, $U \in \mathbb{R}^{N \times N}$ be such that

$$U = [I_N \ U_1], \quad U^{-1} = \begin{bmatrix} \gamma^T \\ U_2 \end{bmatrix}$$

$$U^{-1}G U = \Lambda = \text{diag}(\lambda_1, \cdots, \lambda_N)$$

where $\lambda_i$ for $i = 1, \cdots, N$ are eigenvalues of $G$. Transform $\delta(t)$ with $\xi(t) = (U^{-1} \otimes I_n)\delta(t)$, where $\xi(t) = [\xi_1^T(t), \cdots, \xi_N^T(t)]^T$. Then (17) can be represented as

$$\dot{\xi}(t) = (I_N \otimes A)\xi(t) - (I_N \otimes B_{\text{kin}})\xi(t - \tau_1) + (A \otimes B_{\text{kin}})\xi(t - \tau_2)$$

At the same time, it can be seen that

$$\xi_1(t) = (\gamma^T \otimes I_n)\delta(t) \equiv 0$$

Therefore, $\xi_i(t), i = 2, \cdots, N$ asymptotically converge to zero if and only if the $N - 1$ subsystems

$$\xi_i(t) = A\xi_i(t) - B_{\text{kin}}C\xi_i(t - \tau_1) + \lambda B_{\text{kin}}C\xi_i(t - \tau_2), \quad i = 2, \cdots, N$$

are asymptotically stable.

### 3.1 The characteristic equation

The characteristic function of (5) is given by

$$f(s; \kappa, \tau_1, \tau_2) = \det F(s; \kappa, \tau_1, \tau_2)$$

where the characteristic matrix $F$ is as follows

$$F(s; \kappa, \tau_1, \tau_2) = I_N \otimes (sI_n - A + B_{\text{kin}}e^{-s\tau_1}) - (G \otimes B_{\text{kin}})e^{-s\tau_2}$$
According to $U^{-1}GU = \Lambda$, the characteristic function becomes
\[
f(s; \kappa, \tau_1, \tau_2) = \det(I_N \otimes (sI_n - A + \kappa B C e^{-\tau_1}) - (U U^{-1} \otimes B C) e^{-\tau_2}) = \\
\det(U^{-1} \otimes I_n) \det(I_N \otimes (sI_n - A + \kappa B C e^{-\tau_1}) - (U U^{-1} \otimes B C) e^{-\tau_2}) \det(U \otimes I_n) = \\
\prod_{i=1}^N \det(sI_n - A + \kappa B C e^{-\tau_1}) - (\Lambda \otimes B C) e^{-\tau_2} =
\]
where
\[
f_i(s; \kappa, \tau_1, \tau_2) = \det F_i(s; \kappa, \tau_1, \tau_2) \tag{21}
\]
\[
F_i(s; \kappa, \tau_1, \tau_2) = sI_n - A + \kappa B C e^{-\tau_1} - \lambda_i B C e^{-\tau_2}
\]

\textbf{Remark 3.} The zeros of
\[
f_i(s; \kappa, \tau_1, \tau_2) = \det(sI_n - A + \kappa B C e^{-\tau_1} - \kappa B C e^{-\tau_2})
\]
describe the behavior of the agreement dynamics while the zeros of
\[
f_2(s; \kappa, \tau_1, \tau_2), \ldots, f_N(s; \kappa, \tau_1, \tau_2)
\]
represent the disagreement dynamics of the system. When these disagreement dynamics are stable, the agents will reach consensus.

\textbf{Theorem 1 (Group behavior).} Assume that the communication topology is connected. If the $N-1$ factors $f_2(s; \kappa, \tau_1, \tau_2), \ldots, f_N(s; \kappa, \tau_1, \tau_2)$ are asymptotically stable, then the agents in the network reach a consensus. Furthermore,
\[
x(t) \rightarrow [\eta_1^T(t), \ldots, \eta_N^T(t)]^T
\]
where $\eta_i(t)$ is described by (22) and $\eta_i(0) = (\gamma^T \otimes I_n) x(0)$.

\textbf{Proof.} Let $\eta_i(t) = [\eta_1^T(t), \ldots, \eta_N^T(t)]^T$ and $\eta(t) = (U^{-1} \otimes I_n) x(t)$ where $U$ satisfies $U^{-1} GU = \Lambda = \text{diag}(\lambda_1, \ldots, \lambda_N)$. Then $\eta(t)$ can be described as follows
\[
\dot{\eta}_i(t) = (I_N \otimes \Lambda) \eta_i(t) - (I_N \otimes \kappa B C) \eta(t - \tau_1) + \\
\quad + \kappa (\Lambda \otimes B C) \eta_i(t - \tau_2)
\]
and it can be seen that
\[
\dot{\eta}_i(t) = A \eta_i(t) - \kappa B C \eta_i(t - \tau_1) + \kappa B C \eta_i(t - \tau_2) \tag{22}
\]
\[
\eta_i(t) = A \eta_i(0) - \kappa B C \eta(t - \tau_1) + \kappa B C \eta(t - \tau_2), \quad i = 2, \ldots, N
\]

If $N-1$ factors $f_2(s; \kappa, \tau_1, \tau_2), \ldots, f_N(s; \kappa, \tau_1, \tau_2)$ are asymptotically stable, then $\eta_i(t) \rightarrow [\eta_i^T(t), 0, \ldots, 0]^T$ as $t \rightarrow \infty$. Considering $U = [I_N \ U]$ and taking $x(t) = (U \otimes I_n) x(t)$ into account, it is obtained
\[
x(t) \rightarrow (U \otimes I_n) [\eta_1^T(t), 0, \ldots, 0]^T = [\eta_1^T(t), \ldots, \eta_N^T(t)]^T
\]
where $\eta_i(0) = (\gamma^T \otimes I_n) x(0)$. Therefore consensus is achieved.

\textbf{Remark 4.} The characteristic function $f_1(s; \kappa, \tau_1, \tau_2)$ is related to the agreement behavior of the network. That is, if agents in the network reach consensus, then they will agree in a dynamics described by $f_1(s; \kappa, \tau_1, \tau_2)$. A stable consensus is achieved if $f_1(s; \kappa, \tau_1, \tau_2), i = 1, \ldots, N$ are stable. While, if all $f_i(s; \kappa, \tau_1, \tau_2), i = 2, \ldots, N$ except $f_1(s; \kappa, \tau_1, \tau_2)$ are stable, then the agents will move in coherence but along an unstable trajectory.

The $N$ factors $f_i$ in (21) given by
\[
f_i(s; \kappa, \tau_1, \tau_2) = \det(sI_n - A + \kappa B C e^{-\tau_1} - \kappa \lambda_i B C e^{-\tau_2})
\]
can be rewritten as
\[
f_i(s; \kappa, \tau_1, \tau_2) = s^n + \sum_{j=0}^{n-1} s^j(-a_j + \kappa \lambda_j B C e^{-\tau_2}) \quad \tag{23}
\]
where $a_j, c_j, j = 0, \ldots, n - 1$ are elements of matrices $A$ and $C$, which have the following normalized form
\[
A = \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1 \\
0 & a_0 & a_1 & \cdots & a_{n-1}
\end{bmatrix}, \quad B = \begin{bmatrix}
0 \\
\vdots \\
1
\end{bmatrix}
\]
\[
C = \begin{bmatrix}
c_0 & c_1 & \cdots & c_{n-2} & c_{n-1}
\end{bmatrix}
\]

The characteristic function (23) has infinitely many zeros. If for $i = 1, \ldots, N, \ k$ all these zeros have negative real parts for given delays $\tau_1$ and $\tau_2$, then a stable consensus for all agents would be reached. If the topology of network is fixed, the consensus of the network is reduced to the stability analysis of each factor (23). Then the resulting stability regions are intersected to find combinations in $(\tau_1, \tau_2)$ parametric space that produce stable consensus behaviors for all agents for a fixed value of $\kappa$.

\textbf{3.2 Selection of the control gain}.

For the selection of control gain $\kappa$, it is desired that $N$ factors in (23) are asymptotically stable when delays $\tau_1$ and $\tau_2$ are zero. In delay-free cases, $N-1$ characteristic functions are written as
\[
f_i(s; \kappa) = \det(sI_n - A + \kappa B C - \kappa \lambda_i B C) = \\
s^n + \sum_{j=0}^{n-1} s^j(-a_j + \kappa (1 - \lambda_i) c_j), \quad i = 2, \ldots, N \tag{24}
\]

Let $(1 - \lambda) \kappa = \bar{\lambda}_i > 0, i = 2, \ldots, N$. Consider the polynomial
\[
f(s) = s^n + \sum_{j=0}^{n-1} s^j(-a_j + \sigma c_j), \quad \sigma > 0
\]

It is easy to see that $\tilde{f}(s)$ has zero eigenvalue if and only if $\sigma = \frac{c_0}{a_0} > 0$ for $c_0 \neq 0$. Consider $f(s)$ in (25) where $a_j, c_j, j = 0, \ldots, n - 1$ are constants. As $\sigma$ varies, the sum of the orders of the zeros of $\tilde{f}(s)$ on the open right-half plane can change only if a zero appears on or crosses the imaginary axis. Let $s = j \omega (\omega \neq 0)$. Then $\tilde{f}(s)$ has a purely imaginary root if and only if
\[
(j \omega)^n + \sum_{k=0}^{n-1} (-1)^k (\sigma c_{2k+1} - a_{2k+1}) \theta^k = 0
\]
\[
\sum_{k=0}^{n-2} (-1)^k (\sigma c_{2k+2} - a_{2k+2}) \theta^k + (-1)^{n-1} \theta^2 = 0
\]

Let $\theta = \omega^2 > 0$. If $n = 2 \nu$, where $\nu$ is a positive integer, then separate the real and imaginary parts to obtain
\[
\sum_{k=0}^{\nu-1} (-1)^k (\sigma c_{2k+1} - a_{2k+1}) \theta^k = 0
\]
\[
\sum_{k=0}^{\nu-2} (-1)^k (\sigma c_{2k+2} - a_{2k+2}) \theta^k + (-1)^{\nu-1} \theta^2 = 0
\]
Similarly, if \( n = 2\nu - 1 \), where \( \nu \) is a positive integer, then
\[
\sum_{k=0}^{n-1} (-1)^k (\sigma c_{k+1} - a_{k+1}) \theta^k + (-1)^{\frac{n-1}{2}} \theta^{\frac{n+1}{2}} = 0
\]
(26)

From (26) and (27), one can get positive real values of \( \sigma \) (if there exists) such that \( f(s) \) has a purely imaginary root. Taking \( \sigma = \frac{a_{2\nu}}{c_{2\nu}} \neq 0 \) (if it is positive) into account, suppose that there are \( r \) different positive values, \( 0 = \sigma_0 < \sigma_1 < \cdots < \sigma_r < \sigma_{r+1} = \infty \), such that \( f(s) \) has a purely imaginary root. Let \( \Sigma_j = [\sigma_{j-1}, \sigma_j], j = 1, \ldots, r+1 \).

**Lemma 6.** If there exists a positive \( \sigma \in \Sigma_j \) such that
\[
s^n + \sum_{j=0}^{n-1} s^j(-a_j + c_j) \theta^j = 0
\]
is unstable (stable), then for any \( \sigma \in \Sigma_j \), \( s^n + \sum_{j=0}^{n-1} s^j(-a_j + c_j) \theta^j = 0 \) is unstable (stable), where \( s^n + \sum_{j=0}^{n-1} s^j(-a_j + c_j) \) has a purely imaginary root when \( \sigma = \sigma_{j-1}, k = 1, \ldots, r \).

**Proof.** If \( s^n + \sum_{j=0}^{n-1} s^j(-a_j + c_j) \) is stable (unstable), then all zeros have negative real parts (at least one zero has a positive real part). When \( \sigma = \sigma_{j-1} \) or \( \sigma = \sigma_j \), the polynomial has a purely imaginary zero. The sum of the orders of zeros of polynomial on the open right half plane is unchanged when \( \sigma \in \Sigma_j \).

From Lemma 6, the sum of the orders of zeros of \( f(s) \) on the open right-half plane or left-half plane is unchanged for all \( \sigma \in \Sigma_j \). We define \( \Sigma_j \) as a stable range if, for any \( \sigma \in \Sigma_j \), \( s^n + \sum_{j=0}^{n-1} s^j(-a_j + c_j) \theta^j = 0 \) is stable. Therefore, \( N-1 \) characteristic equations \( f(s; \kappa) = 0 \), where \( f(s; \kappa) \) is defined in (24), are asymptotically stable if \( \lambda_i \in \Sigma_j, \ i = 2, \ldots, N \).

**Lemma 7.** Suppose that \( \Sigma_j \) is a stable range for some \( j = 1, \cdots, r+1 \). If \( \frac{1}{\rho_{\text{min}}} \frac{\lambda_{\text{max}}}{\lambda_{\text{min}}} < \frac{\sigma_j}{\sigma_{j-1}} \), then there exists a value \( \kappa > 0 \) such that \( \frac{1}{\rho_{\text{min}}} \frac{\lambda_{\text{max}}}{\lambda_{\text{min}}} < \kappa < \frac{1}{\rho_{\text{min}}} \frac{\lambda_{\text{max}}}{\lambda_{\text{min}}} \) and \( f(s; \kappa) = 0 \) for all \( i = 2, \ldots, N \) are asymptotically stable. Furthermore, if matrix \( A \) is also Hurwitz stable, then \( N \) factors \( f(s; \kappa) \) in (23) are asymptotically stable when delays \( \tau_1 \) and \( \tau_2 \) are zero.

**Proof.** If there exists a \( \kappa > 0 \) such that \( \frac{1}{\rho_{\text{min}}} \frac{\lambda_{\text{max}}}{\lambda_{\text{min}}} > \sigma_{j-1} \) and \( \frac{1}{\rho_{\text{min}}} \frac{\lambda_{\text{max}}}{\lambda_{\text{min}}} < \kappa \), then it completes the proof. If the condition \( \frac{1}{\rho_{\text{min}}} \frac{\lambda_{\text{max}}}{\lambda_{\text{min}}} < \kappa < \frac{1}{\rho_{\text{min}}} \frac{\lambda_{\text{max}}}{\lambda_{\text{min}}} \) holds, then \( \frac{c^j(\sigma_j - \kappa)}{\sigma_j - \kappa} \) for acquiring \( \Omega^\nu \) through (12). According to Lemma 6, we choose to investigate the zeros of \( R_{\Omega}^\nu(\theta_{\text{fin}}, \tau_{\text{fin}}) \) instead of studying those of \( g_\Omega(\omega; T_1, T_2) \) for acquiring \( \Omega^\nu \).

**3.3 Stability analysis.**

In this subsection, we deploy the advanced clustering with frequency sweeping method to analyze the stability of \( N \) factors (23). And we assume that \( \kappa \) is chosen from stable ranges as stated in Remark 6. The goal is to find out how delay terms can affect the consensus behavior of the system, i.e., what variations of delay terms could be tolerated for stable consensus when all parameters except delays are fixed.

First of all, all critical delay pairs \( (\tau_1, \tau_2) \) which generate stability switching need to be found out. In order to attain this, for each individual factor \( f_i(s; \tau_1, \tau_2), i = 1, \cdots, N \) in (23), the procedure of applying ACSF method is as follows.

1) \( f_i(\omega; \tau_1, \tau_2) \) in (23) can be transformed to \( g_i(\omega; T_1, T_2) \) by using the Rekasius transformation (8). Thus the crossing frequency set \( \Omega^\nu \) defined by \( f_i(\omega; \tau_1, \tau_2) = 0 \) and the crossing frequency set \( \Omega^\nu \) by \( g_i(\omega; T_1, T_2) = 0 \) have the relationship \( \Omega^\nu = \Omega^\nu \) by Lemma 3. In order to obtain \( \Omega^\nu \), we turn to finding \( \Omega^\nu \) through \( g_i(\omega; T_1, T_2) = 0 \).

2) Further, by partitioning \( g_i(\omega; T_1, T_2) \) into the real part \( \text{Re}(g_i(\omega; T_1, T_2)) \) and imaginary part \( \text{Im}(g_i(\omega; T_1, T_2)) \), the resultant \( R_{\Omega}^\nu(\text{Re}(g_i(\omega; T_1, T_2)), \text{Im}(g_i(\omega; T_1, T_2))) \) can be easily computed through (12). According to Lemma 4, we choose to investigate the zeros of \( R_{\Omega}^\nu(\text{Re}(g_i), \text{Im}(g_i)) \) instead of studying those of \( g_i(\omega; T_1, T_2) \) for acquiring \( \Omega^\nu \).

3) By defining \( R_{\Omega}^\nu(\text{Re}(g_i), \frac{\partial R_{\Omega}^\nu}{\partial \omega}) \) according to (12), where \( R_{\Omega}^\nu(\text{Re}(g_i), \frac{\partial R_{\Omega}^\nu}{\partial \omega}) \) is the resultant of \( R_{\Omega}^\nu(\text{Re}(g_i), \text{Im}(g_i)) \), the lower bound \( \omega^\nu \) and upper bound \( \omega^\nu \) of \( \Omega^\nu \) could be determined through Lemma 5.

Secondly, based on the resulted lower and upper bounds \( \omega^\nu \) and \( \omega^\nu \), we have the following conclusion of delay-independent consensus of (5).

**Theorem 2.** Assume that the communication topology is connected. The consensus of all agents will be delay-independently achieved if and only if

1) System in (5) is asymptotically stable when delays \( \tau_1 \) and \( \tau_2 \) are zero.

2) For \( i = 1, \cdots, N \), \( R_{\Omega}^\nu(\text{Re}(g_i), \frac{\partial R_{\Omega}^\nu}{\partial \omega}) = 0 \) have no positive real zeros that give rise to \( (T_1, T_2) \in \mathbb{R}^2 \) solutions.

**Proof.** The second condition indicates that \( \Omega^\nu \) is empty for \( i = 1, \cdots, N \). Since \( \Omega^\nu = \Omega^\nu \) and \( \Omega^\nu \) generates the stability switches, the condition that \( \Omega^\nu \) is empty does not produce stability changes, and vice versa, in the delay parametric space. Thus the behaviors of system are conducted the same as that of delay-free system. Together with the first condition, it yields that each \( f_i(s; \tau_1, \tau_2), i = 1, \cdots, N \) in (23) is stable and thus the delay-independent consensus of all agents is achieved according to Theorem 1.

Thirdly, if Condition 1 holds in Theorem 2, but Condition 2 is not satisfied, we investigate the problem of the delay-dependent consensus of all agents. In other words, tolerances for variations of delay terms are desired to be decreased for the achievement of stable consensus behavior.

Through Lemma 5, for \( i = 1, \cdots, N \), the lower bound \( \omega^\nu \) and upper bound \( \omega^\nu \) of \( \Omega^\nu \) can be computed by \( R_{\Omega}^\nu(\text{Re}(g_i), \frac{\partial R_{\Omega}^\nu}{\partial \omega}) = 0 \). Based on \( \omega^\nu \) and \( \omega^\nu \), the following algorithm aims to compute stability/instability regions of system (5) in the delay parametric space.

**Algorithm 1.** 1) Repeat for \( i = 1, \cdots, N \). For each \( \omega \in [\omega^\nu, \omega^\nu] \), choose an appropriate step size and perform the following steps:

a) Solve the equation \( R_{\Omega}^\nu(\text{Re}(g_i), \text{Im}(g_i)) = 0 \) to obtain real
values of $T_1$, where $R_{T_1}^2 (\bar{g}_{th}, \bar{g}_{lm})$ can be computed according to (12).

b) For each real value $T_1$ found from the previous step a), if there exists real values $T_2$ satisfying $\bar{g}_{th} = 0$ and $\bar{g}_{lm} = 0$, then proceed to the next step, otherwise increase $\omega$ with step size, and restart from step a).

c) Through (11), compute delays $(\tau_1, \tau_2)$ corresponding to $(T_1, T_2)$ found in previous two steps. Increase $\omega$ with step size and restart from step a).

2) Combine the obtained results for $i = 1, \ldots, N$. This yields a full characterization of the stability regions of (5) in the delay parametric space, because all critical delay values are covered.

Remark 7. For a specified $\omega^*$, it is possible to encounter the first three cases mentioned in Lemma 4. If 1) appears, then compute real $T_1, T_2$ values through $\bar{g}_{lm} = 0$ when $\omega = \omega^*$. Similarly, if 2) appears, then compute real $T_1, T_2$ values through $\bar{g}_{th} = 0$. If 3) appears for a given $\omega^*$, then it indicates that $T_2 \to \infty$.

Remark 8. Algorithm 1 only sweeps the delay parameters for a fixed value of control gain $\kappa$. As stated in Remark 6, $\kappa$ selected from stable ranges guarantees the achievement of consensus in delay-free case. For a fixed $\kappa$, compositions of two delays that locate in stable regions resulted from Algorithm 1 represent the allowable variations of delays, that is, the compositions of two delays in stable regions produce delay-dependent consensus of agents.

Remark 9. Theorem 2 gives sufficient and necessary conditions for the delay-independent consensus of all agents. And Algorithm 1 depicts compositions of two delays which produce delay-dependent consensus of agents. For both cases, the network of agents reaches a consensus. And Theorem 1 shows the group behavior of the network.

4 Example

Consider a network consisting of 20 agents with communication topology in Fig. 1. Assume that weights in the adjacency matrix $A(G)$ satisfy that $a_{ij} = a_{ji} = 1$ if $i$-th agent and $j$-th agent ($i \neq j$) are connected, otherwise $a_{ij} = a_{ji} = 0$. The eigenvalues of matrix $G$ are listed in Table 1.

![Fig.1 Topology structure of the network with 20 agents](image)

Matrices $A, B, C$ are assumed to be in the normalized forms and elements of matrices $A, C$ are as

$$
a_{00} = -10, \quad a_{11} = -1.3, \quad c_0 = -1, \quad c_1 = 0.
$$

For delay-free characteristic functions (24), the stable ranges of $(1 - \lambda_i)$ such that $f_i(s; \kappa) = 0$ for $2, \ldots, N$ are asymptotically stable is $S_1 = (0, 10)$. Therefore, $\kappa$ should be chosen in the ranges $(0, 5, 2369)$. In the following, $\kappa = 4$ is selected. It is obtained that system (5) is asymptotically stable when delays $\tau_1$ and $\tau_2$ are zero.

When $\kappa = 4$, the procedures to check the delay-independent consensus of all agents by the advanced clustering with frequency sweeping method are as follows.

First, the Rekasius transformation (8) is used to convert (23) to (9). Then (9) is separated into two parts as (14) and (15). The resultant $R_{T_1}^2 (\bar{g}_{th}, \bar{g}_{lm})$ is used to eliminate the parameter $T_2$. According to obtained $R_{T_1}^2 (\bar{g}_{th}, \bar{g}_{lm})$, compute $\partial R_{T_1}^2 / \partial T_1$. For two polynomials $R_{T_2}^2 (\bar{g}_{th}, \bar{g}_{lm})$ and $\partial R_{T_1}^2 / \partial T_1$ with respect to variables $\omega$ and $\kappa$, the resultant $R_{T_1}^2 (\bar{g}_{th}, \bar{g}_{lm})$ is applied to eliminate the parameter $T_1$. The above procedures lead to a univariate polynomial with respect to $\omega$. Solve zeros of $Z'(\omega) = R_{T_1}^2 (\bar{g}_{th}, \bar{g}_{lm}) = 0$ and find that positive real zeros of $Z'(\omega)$ exist for all $i = 1, \ldots, N$. For each $\Omega_1$, the lower bound $\omega^*$ and upper bound $\omega^{*'}$ of $\Omega_1$ are listed in Table 1. For all $\omega^*$ and $\omega^{*'}$, by back substitution of these values into $R_{T_1}^2 = 0$, the resultant $R_{T_1}^2 (\bar{g}_{th}, \bar{g}_{lm}) = 0$, (14) and (15), compute the corresponding $T_1, T_2$, which are found to be real numbers. So far, we could conclude that the delay-independent consensus of all agents cannot be reached.

### Table 1: Eigenvalues of $G$, lower and upper bounds of $\Omega^i$

<table>
<thead>
<tr>
<th>$i$</th>
<th>$\lambda_i$</th>
<th>$\omega^*$</th>
<th>$\omega^{*'}$</th>
<th>$\lambda_i$</th>
<th>$\omega^*$</th>
<th>$\omega^{*'}$</th>
<th>$\lambda_i$</th>
<th>$\omega^*$</th>
<th>$\omega^{*'}$</th>
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<td>3.9027</td>
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<td>3.8491</td>
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<td>1.9949</td>
<td>3.7855</td>
<td>-0.6346</td>
<td>2.2501</td>
<td>3.6397</td>
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</tr>
<tr>
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<tr>
<td>12</td>
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<td>3.3476</td>
<td>-0.01</td>
<td>2.7155</td>
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<td>4.0087</td>
<td>1.00</td>
<td>1.4967</td>
<td>4.0087</td>
</tr>
</tbody>
</table>

Subsequently, the delay-dependent consensus of agents is discussed and the stability regions in delay parametric space are obtained. The algorithm described in previous section is performed with step size as 0.001. For each factor $f_i(s; \kappa, \tau_1, \tau_2)$, upon sweeping $\omega$ in the range $(\omega^*, \omega^{*'})$, the stability charts in the delay parametric space are created in Fig. 2. The lines in Fig. 2 exhaustively represent $(\tau_1, \tau_2)$ combinations, for which $f_i(s; \kappa, \tau_1, \tau_2) = 0$ has a purely imaginary root.

We can see from Fig. 2 that stability boundaries consist of a series of closed curves clustering in groups. Every group includes five curves and each of them represents the stability boundary of one factor. Fig. 2(a) describes the cases for $\lambda = \{\lambda_{10}, \lambda_{19}, \lambda_{18}, \lambda_{17}, \lambda_{16}\}$, and in each group the curves from the outermost to the innermost correspond to $\lambda_{16}$ to $\lambda_{10}$; Fig. 2(b) for $\lambda = \{\lambda_{15}, \lambda_{14}, \lambda_{13}, \lambda_{12}, \lambda_{11}\}$, and the curves from the outermost to the innermost correspond to $\lambda_{11}$ to $\lambda_{15}$ respectively. Fig. 2(c) for $\lambda = \{\lambda_{10}, \lambda_{9}, \lambda_{8}, \lambda_{7}, \lambda_{6}\}$, and the curves from the outermost to the innermost correspond to $\lambda_{10}$ to $\lambda_{6}$ respectively; Fig. 2(d) for $\lambda = \{\lambda_{5}, \lambda_{4}, \lambda_{3}, \lambda_{2}, \lambda_{1}\}$, and the curves from the outermost to the innermost correspond to $\lambda_{5}$ to $\lambda_{1}$ respectively. From Fig. 2 it is seen that the factors corresponding to the maximal and minimal values of $\lambda_i, i = 1, \ldots, N$ introduce the most restrictive stability regions. When the individual stability plots for all $i = 1, \ldots, N$ are superimposed and intersected, the stability region for the complete system can be identified as the
shaded regions in Fig. 3. Furthermore, considering $\tau_1 = \tau_{in}$ and $\tau_2 = \tau_{in} + \tau_{com}$, by transforming the $(\tau_1, \tau_2)$ space to $(\tau_{in}, \tau_{com})$ space, the stability regions in $(\tau_{in}, \tau_{com})$ space can be obtained as the shaded regions in Fig. 4.

All $(\tau_1, \tau_2)$ compositions within the shaded regions in Fig. 3 produce a stable delay-dependent consensus of all agents. While points located outside these regions result in unstable behaviors, that is, consensus cannot be reached. Fig. 5 describes stable state behaviors of five agents with $(\tau_1, \tau_2) = (0, 0.5)$ corresponding to point a in Fig. 3. While Fig. 6 presents unstable state behaviors with $(\tau_1, \tau_2) = (1.6, 2.2)$ corresponding to point b in Fig. 3. The number $i = \{1, 5, 10, 15, 20\}$ close to curves in Fig. 5 and Fig. 6 indicates the $i$-th agent.

Fig. 2 Stability boundaries of individual factors

Fig. 3 Stability regions (shaded) in $(\tau_1, \tau_2)$ space

Fig. 4 Stability regions (shaded) in $(\tau_{in}, \tau_{com})$ space
Fig. 5 Stable responses of five agents with point a in Fig. 3

Fig. 6 Unstable responses of five agents with point b in Fig. 3

The next problem is the convergence speed of consensus. The speed relies on the rightmost root of the system. It can be seen that the rightmost root of $f_i(s, \kappa, \tau_1, \tau_2)$ for $i = 1, \cdots, N$ determines the speed. For point a in the example, the real part of the rightmost root is $-0.3178$. Correspondingly, a settling time of approximately four times the consensus speed is 12.59 s as observed in Fig. 5.

5 Conclusions

In this paper, we discuss the consensus problem of a group of general linear agents under an undirected topology. Both input and communication delays are taken into account. The factorization of the characteristic equation of the system into decreased-order factors relies only on the set of eigenvalues of a matrix that describes the structure of the network topology, and simplifies the stability analysis considerably. The influence of control gain is considered and we restrict it to the stable ranges within which the consensus of system could be reached when delays vanish. For a fixed control gain chosen from stable ranges, the advanced clustering with frequency sweeping method is used to determine the achievement of delay-independent/delay-dependent consensus of all agents, and also to obtain stability bounds in delay parametric space.

References


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