Wavelet Inpainting Based on Tensor Diffusion

YANG Xiu-Hong1 GUO Bao-Long1 WU Xian-Xiang1

Abstract Due to the lossy transmission in the JPEG2000 image compression standard, the loss of wavelet coefficients heavily affects the quality of the received image. In this paper, we propose a novel wavelet inpainting model based on tensor diffusion (TDWI) to restore the missing or damaged wavelet coefficients. A hybrid model is built by combining structure-adaptive anisotropic regularization with wavelet representation. Its associated Euler-Lagrange equation is also given for analyzing its regularity performance. Owing to the matrix representation of the structure tensor in the regularization term, the shape of diffusion kernel changes adaptively according to the image features, including sharp edges, corners and homogeneous regions. Compared with existing wavelet inpainting models, the proposed one can control more adaptively and accurately the geometric regularity in the image and exhibits better robustness to noise. In addition, an effective and proper numerical scheme is adopted to improve the computation. Experimental results on a variety of loss scenarios are given to demonstrate the advantages of our proposed model.

Key words Wavelet, inpainting, structure tensor, diffusion, anisotropic, regularization


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Image inpainting refers to as filling in missing or damaged information in an image. It plays an important role in computer vision and image processing, including image replacement[1], disocclusion[2–3], and error concealment[4–6].

The missing information can occur in the pixel domain, as well as the transformed domain. After the release of the image compression standard JPEG2000, wavelet transform plays an important role in image formatting, storing, compression and transmission coding. During the lossy storing or transmission, the loss of wavelet packets in received sub-bands may heavily affect the image quality. Therefore, it is necessary to recover the source image by using the incomplete wavelet data, which can be addressed as wavelet inpainting originally proposed in [7], since it is closely related to the classic image domain inpainting.

However, wavelet inpainting differs from its image domain counterpart. For image domain inpainting, the clear cut inpainting regions are easy to be localized and various kinds of diffusion schemes based on variational partial differential equation (PDE) models can be directly used to restore the damaged regions. For wavelet inpainting, however, the loss of wavelet coefficients in some subbands affects the whole image quality. Therefore, the resulting damaged regions cannot be localized simply in the pixel domain. This challenge prohibits the direct application of the existing PDE models[8–9] to wavelet inpainting. Besides, the property of wavelet decoupling makes it insufficient to directly interpolate in the wavelet domain. Chan et al.[10] proposed two related variational models for wavelet inpainting, which combine the total variation (TV) minimization technique with wavelet representation. Their main idea is to use TV regularization in the pixel domain to control and restore the wavelet coefficients in the wavelet domain. The main benefit of the TV regularization is that it can preserve edges very well[8], but it suffers from the staircase artifact. Aiming to overcome the defect, Zhang et al.[10] adopted the p-Laplace operator which gives better diffusion performance. Yau et al.[11] took the L0-norm in the wavelet domain and then filled the missing information by the TV minimization. Hadji et al.[12] used a tixotrop model to make use of the smoothing model and correct the lying out of the contours by putting them more clearly. Zhang et al.[13] used a nonlocal TV operator to extend the Chan, Shen and Zhou’s TV wavelet inpainting models.

Although existing regularization methods are quite efficient for recovering damaged wavelet coefficients as presented in [7, 10–13], it is well known that these regularization terms only use local gradient magnitude as a measure to control diffusion performance without using the direction information of the structures in the image. In this paper, we explore the feasibility of structure-adaptive anisotropic regularization to exploit both structure magnitude and direction information in wavelet inpainting. The anisotropic structure tensor uses the matrix representation of the gradient, thus we can accurately estimate the strength of structures and the direction of the maximum variation in the image[14–16]. Accordingly, we can design a matrix-valued diffusion tensor to control the amount of diffusion and the orientation of smoothing. Therefore, instead of using local gradient based regularizations, we use the structure-adaptive anisotropic regularization to control the geometric regularity in the pixel domain to restore the damaged wavelet coefficients.

Our main contribution in this paper is to present a wavelet inpainting model based on tensor diffusion and build up its associated Euler-Lagrange equation to analyze its structure-adaptive anisotropic regularization. The proposed model is solved by an efficient scheme. The advantages of this effective model for different loss cases are demonstrated by a large number of numerical simulations.

This paper is organized as follows. In Section 1, we propose the TDWI model. In Section 2, according to its associated Euler-Lagrange equation, we compare the performance of the TDWI model with other wavelet inpainting models. Section 3 gives the implementation scheme for the proposed model. Generic numerical examples in Section 4 further highlight the remarkable inpainting qualities of the proposed model. The Appendix A gives the derivation of the Euler-Lagrange equation of the proposed model.

1 Wavelet inpainting model based on tensor diffusion

A standard image model is given by

\[ u(x) = f(x) + n(x) \]  (1)
satisfies \( \Psi \) direction of their corresponding eigenvectors. The eigenvalues specify the variations of the image in the local analysis of image structures can give accurate smooth the tensor data.

\[
\lambda_j k = \{ \lambda_j, (j,k) \in \Omega, (j',k') \in I \}
\]

where parameter \( \lambda \) is a positive constant and \( \text{tr}(\cdot) \) is the trace operator (sum of the diagonal elements).

In (4), \( J \) is the tensor structure which is defined as

\[
J = \nabla u \nabla u^T = \begin{bmatrix} u_{11} & u_{12} \\ u_{12} & u_{22} \end{bmatrix} = \begin{bmatrix} (u_s)^2 & u_s u_y \\ u_s u_y & (u_y)^2 \end{bmatrix}
\]

where the gradient \( \nabla u = (u_x, u_y)^T \). Since \( J \) is easily polluted by noise, we could use proper filtering techniques to smooth the tensor data.

\( J \) is a symmetric positive semidefinite matrix. Hence, there exist two nonnegative eigenvalues \( \mu_1 \) and \( \mu_2 \) (setting \( \mu_1 \geq \mu_2 \)) and the corresponding mutually orthogonal eigenvectors \( v_1 \) and \( v_2 \). Thus, \( J \) can also be expressed as

\[
J = \mu_1 v_1 v_1^T + \mu_2 v_2 v_2^T
\]

Its eigenvalues specify the variations of the image in the directions of their corresponding eigenvectors. The eigenvector \( v_1 \) is orthogonally aligned to the edges of an image, while the eigenvector \( v_2 \) determines the dominant orientation of the local structure. Therefore, \( J \) can give accurate local analysis of image structures.

In (4), \( \Psi(s^2) \) is a penaliser which is a differentiable and increasing function that is convex in \( s \) and its derivative satisfies \( \Psi'(s^2) = g'(s^2) \), so we choose it as

\[
\Psi(s^2) = \varepsilon s^2 + (1-\varepsilon) \gamma^2 \left( 1 + \frac{s^2}{\gamma^2} \right)
\]

Correspondingly,

\[
g(s^2) = \varepsilon + (1-\varepsilon) \left( 1 + \frac{s^2}{\gamma^2} \right)
\]

where \( 0 < \varepsilon < 1, \gamma > 0 \), thus, function \( g(s^2) \) must be a decreasing function.

Both \( \Psi(s^2) \) and \( g(s^2) \) are scalar-valued functions, and then we generalize them to matrix-valued functions as follows:

\[
\Psi(J) = \sum \Psi(\mu_i) v_i v_i^T
\]

\[
g(J) = \sum g(\mu_i) v_i v_i^T
\]

2 Analysis of the proposed model

For the energy functional \( F(u,f) \) in (4), the corresponding Euler-Lagrange equation (which is deduced in Appendix A) is given by

\[
- (\nabla \cdot (g(J) \nabla u(\beta, x)) \cdot \psi_{j,k}) + \sum \lambda_{j,k} (\beta_{j,k} - \alpha_{j,k}) = 0
\]

where \( \nabla \) denotes the divergence operator, and \((,\cdot)\) denotes the inner product.

The corresponding gradient descent flow of (11) with an artificial time variable \( \tau \) is

\[
(\beta_{j,k})_\tau = (\nabla \cdot (g(J) \nabla u(\beta, x)) \cdot \psi_{j,k}) - \sum \lambda_{j,k} (\beta_{j,k} - \alpha_{j,k})
\]

The first term of (12) combines structure-adaptive anisotropic regularization with wavelet decomposition. The regularity in the pixel domain is dependent on the term below:

\[
\Delta_t u = (J \nabla u(\beta, x)) = (D \nabla u(\beta, x))
\]

(13) where \( D = g(J) \) is referred to as a \( 2 \times 2 \) diffusion tensor. According to (10), we can obtain that

\[
D = g(J) = \sum g(\mu_i) v_i v_i^T = \sum k_i v_i v_i^T
\]

where \( k_1 = g(\mu_1) \) and \( k_2 = g(\mu_2) \). Therefore, \( (k_1, v_1) \) and \( (k_2, v_2) \) are the eigenpairs of the diffusion tensor \( D \). As in Fig. 1 (a), \( v_1 = (\cos \theta, \sin \theta)^T \) and \( v_2 = (\sin \theta, \cos \theta)^T \), where \( \theta \) is the direction of \( v_1 \).

(a) The shape of the kernel (b) The kernels in different regions

Fig. 1 The Gaussian kernels

Since the diffusion process with diffusion time \( \tau \) equals the Gaussian convolution with standard deviation \( \rho = \sqrt{2\tau} \), we can get the corresponding Gaussian convolution kernel of (13) for pixel \( x \) as

\[
\begin{cases}
\begin{aligned}
u(x, \tau) &= u(x, 0) \otimes G_{\rho = \sqrt{2\tau}} \\
G_{\rho = \sqrt{2\tau}} &= \frac{1}{4\pi \tau} \exp \left( \frac{x^T D^{-1} x}{4\tau} \right)
\end{aligned}
\end{cases}
\]

(15)
where denotes the convolution operator.

According to (14), we know that is also a symmetric and positive semidefinite matrix, then we obtain

\[
D^{-1} = \sum_i \frac{1}{k_i} v_i v_i^T = \begin{bmatrix}
\frac{\cos^2\theta + \sin^2\theta}{k_1} & \sin\theta\cos\theta \left(\frac{1}{k_1} - \frac{1}{k_2}\right) \\
\sin\theta\cos\theta \left(\frac{1}{k_1} - \frac{1}{k_2}\right) & \frac{\sin^2\theta + \cos^2\theta}{k_2}
\end{bmatrix}
\]

(16)

where \(\{v_1, v_2\}\) are the eigenvectors of \(D^{-1}\).

Thus,
\[
x^T D^{-1} x = \frac{y_1^2}{k_1} + \frac{y_2^2}{k_2}
\]

(17)

Since \(D^{-1}\) is a real symmetric matrix, there exists the orthogonal transformation of (17). Therefore, we can get its quadratic form according to its normalized form that

\[
x^T D^{-1} x = \frac{y_1^2}{k_1} + \frac{y_2^2}{k_2}
\]

(18)

Thus, the Gaussian kernel \(G_p\) is

\[
G_p|_{p=\sqrt{\kappa}} = \frac{1}{4\pi\tau} \exp\left(\frac{-y_1^2 + y_2^2}{4\tau}\right)
\]

(19)

As shown in Fig. 1, we can analyze the shape of the Gaussian kernel according to the local structure feature in the image:

1) When \(x\) belongs to the regions with a linear structure, we can obtain that \(\mu_1 \gg \mu_2 \approx 0\) and \(k_1 \ll k_2\). Thus the shape of the Gaussian kernel is a highly oriented ellipse whose main axis is parallel to the dominant orientation \(v_2\), which leads to structure-adaptive anisotropic regularization and allows to smooth along image edges while inhibiting smoothing across edges.

2) When \(x\) belongs to the homogeneous regions, we can obtain that \(\mu_1 \approx \mu_2 \approx 0\) and \(k_1 \approx k_2 \approx 0\). Thus the shape of the Gaussian kernel is a large circular, which leads to a strong isotropic regularization.

3) When \(x\) belongs to the regions with a corner or junction, we can obtain that \(\mu_1 \approx \mu_2 \gg 0\) and \(k_1 \approx k_2 \approx 0\). Thus the kernel’s shape is a small circular, which leads to a weak isotropic regularization.

Therefore, the regularization property depends on the Gaussian kernel whose shape changes adaptively according to the local structure feature. Besides, the integration of local orientation makes it applicable to distinguishing areas with edges from areas with corners. Hence, the proposed model can even perform adaptive regularization at the presence of corners.

Next, we analyze the physical characteristics of other anisotropic diffusion operators which also consider the structure information, such as TV regularization in [7–8], the \(p\)-Laplace regularization in [10], nonlocal TV regularization in [13] and the \(p\)-CDD operator in [9].

\[
\Delta_{TVu} = \nabla \cdot (|\nabla u|^{-2} \nabla u) = |\nabla u|^{-1} u_{\xi\xi}
\]

(20)

\[
\Delta_{\nu} = \nabla \cdot ((\nabla u)^{p-2} \nabla u) = |\nabla u|^{p-2} u_{\xi\xi} + (p-1) |\nabla u|^{p-2} u_{\eta\eta}
\]

(21)

\[
\Delta_{NLTV} = \nabla \cdot (|\nabla u|^{-1} \nabla u)
\]

(22)

where \(\xi = \nabla^2 u/|\nabla u|\) and \(\eta = |\nabla u|/|\nabla u|\). In the local orthogonal coordinates \((\xi, \eta)\), as shown in Fig. 2, the \(\eta\)-axis is parallel to the gradient direction and the \(\xi\)-axis is perpendicular.

\[
\Delta_{p,CDD} u = \nabla \cdot \left(\frac{\Delta_p u}{|\nabla u|} \nabla u\right)
\]

(23)

Although the geometric information is additionally taken into account to decide the diffusion strength, the diffusivity \(|\nabla u|/|\nabla u|\) is also scalar-valued as the same as (20) – (22). It does not utilize the direction information of the structure to control the diffusion process either.

Compared with the four regularizations in (20) – (23), the structure-adaptive anisotropic regularization adopted in the proposed model utilizes the matrix-valued diffusion tensor to exploit more structure information. It can adaptively control the geometric regularity in the pixel domain according to both structure strength and structure direction. The matrix representation of the structure tensor allows the integration of information from a local neighborhood, which leads to better robustness to noise. Thus, its estimations of the diffusion orientation and diffusion amount are more accurate and robust.

3 Implementation scheme

Instead of the unconstrained model (4), we consider its constrained version as

\[
\beta^* = \arg\min_u \left\{ \frac{1}{2} \int_{\Omega} \text{tr} (\Psi (J)) \, dx \right\}
\]

s.t. \(\|P_t (\beta) - a\|^2 \leq \varepsilon\)

(24)
where $P_I(\beta) = \left\{ \begin{array}{ll} \beta_{j,k}, & (j,k) \in \Omega \backslash I \\ 0, & \text{else} \end{array} \right.$.

Let $A = P_I W$, where $W$ denotes the forward wavelet transform. By Bregman iteration, problem (24) can be solved by the following iteration scheme:

$$\begin{align*}
\{ u_{n+1} \} = \arg \min_u & \left\{ \frac{1}{2} \int_{\Omega} (\Psi(J)) \, dx \right\} + \frac{1}{2} \| A u - \beta_n \|^2 \\
\beta^{n+1} = \beta^n + (\beta - A u^{n+1})
\end{align*}$$

(25)

for a positive parameter $\mu > 0$. Here, $\beta^n = \beta$ for noise-free case.

The first subproblem is solved by the proximal forward-backward splitting (PFBS) algorithm as used in [13]. The idea of PFBS is to solve a sum of two convex functionals by one-step forward gradient descent on one functional and one-step backward inverting on another functional. Let $u^{n+1,0} = u^n$ and for $t \geq 0$,

$$u^{n+1,t+1} = \arg \min_u \left\{ \frac{1}{2} \int_{\Omega} (\Psi(J)) \, dx \right\} + \frac{1}{2\mu t} \| u - \left( u^{n+1,t} - \eta A^T (A u^{n+1,t} - \beta^n) \right) \|^2$$

where $\eta$ is a positive parameter and satisfies $0 < \eta < 2/\| A^T A \|$. $A^T = W^T J^T$, where $J^T$ is the zero-padding operator and $W^T$ can be implemented by the inverse wavelet transform $W^{-1}$.

Finally, problem (4) can be solved by the two-step iteration scheme as follows:

$$\begin{align*}
v_{n+1,t+1} = & u^{n+1,t} - \eta W^{-1} P_I^T (P_I W u^{n+1,t} - \beta^n) \\
u_{n+1,t+1} = & \arg \min_u \left\{ \frac{1}{2} \int_{\Omega} (\Psi(J)) \, dx \right\} + \frac{1}{2\mu t} \| u - v^{n+1,t+1} \|^2
\end{align*}$$

(27a)

(27b)

(27c)

(27d)

(27e)

(27f)

(28)

Let $U = u^{n+1,t+1}$. (28) can be solved by the following non-linear PDE:

$$\frac{\partial U}{\partial t} = \nabla \cdot (g(J) \nabla u) - \frac{1}{\mu t} (u - V)$$

(29)

We use the following PDE to obtain the smoothed tensor data $J$ as in [16],

$$\begin{align*}
\frac{\partial u_{i,j}}{\partial t}(x,r) = & \nabla \cdot (g(R_e) \nabla u_{i,j}) \quad i, j = 1, 2 \\
\end{align*}$$

(30)

where the diffusivity matrix $R_e$ is obtained by the convolution of each channel of the initial tensor with the Gaussian kernel $G_\sigma$. We use a simple explicit scheme to solve the PDE and the time step size is denoted by $\Delta_t$.

Algorithm 1.

1) Start with $n = 0$, initial guess $\beta^{n=0} = \sigma_{j,k} \chi_{j,k}$, $\chi_{j,k} = \left\{ \begin{array}{ll} 0, & (j,k) \in I \\
1, & (j,k) \notin I \end{array} \right.$

Set $\beta^{n=0} = \beta^{n=0}$;

2) While $n \leq N_{\text{outer}}$ and not stopping criterion, do

a) $\beta_{\text{old}} = \beta_{\text{new}}$;

b) $J^0 = (\nabla u^0, \nabla v^0)^T = \begin{bmatrix} u_{11}^0 & u_{12}^0 \\
u_{21}^0 & u_{22}^0 \end{bmatrix}$

and

$$R_e = G_\sigma \otimes J^0.$$

c) for $m = 1 : N_{\text{inner}}$

i) $P_{pq}^m = \nabla : (g(R_e) \nabla u_{pq}^{m-1})$, $p, q = 1, 2$

which satisfies the boundary condition:

$$g(R_e) \nabla u_{pq}^{m-1} \cdot \mathbf{n} = 0$$

on $\partial \Omega$, where $\mathbf{n}$ is the normal vector on $\partial \Omega$.

ii) $u_{pq}^m = u_{pq}^{m-1} + \Delta_t P_{pq}^m$, $p, q = 1, 2$.

iii) $J = \begin{bmatrix} u_1^m & u_2^m \\
u_1^m & u_2^m \end{bmatrix}$

end the for loop.

d) for $l = 0 : N_{\text{inner}}$

i) $u^{n+1,l} = u_{\text{old}}$.

ii) $v^{n+1,l+1} = u^{n+1,l} - \eta W^{-1} P_I^T (P_I W u^{n+1,l} - \beta_{\text{old}})$.

iii) Let $V = v^{n+1,l+1}$, solving $u^{n+1,l+1}$ using (29)

$$u^{n+1,l+1} = \arg \min_u \left\{ \frac{1}{2} \int_{\Omega} (\Psi(J)) \, dx \right\} + \frac{1}{2\mu \eta t} \| u - V^{n+1,l+1} \|^2$$

end the for loop.

e) new image $u^{n+1} = u^{n+1,l_{\text{inner}}}$.

f) update $\beta_{\text{new}} = \beta_{\text{old}} + \beta^0 - P_I W u^{n+1}$, $n = n + 1$.

end the while loop.

For noise-free case, the iteration scheme should be taken as $\beta_{\text{old}} = \beta^0$ in (2d)(ii) step above and the stopping criterion is

$$\| P_I W u - \beta^0 \| < 10^{-2}.$$ For noisy case, the stopping criterion is $E \leq 10^{-2}$.

4 Experimental results

In this section, we simulate the proposed algorithm on Matlab R2009b. In all simulations, we use Daubechies 7/9 biorthogonal wavelet with symmetric extension, which is used in the standard JPEG2000 for lossy compression and transmission. The standard peak signal to noise ratio (PSNR) as shown in (31) is employed to objectively evaluate the proposed model:

$$\text{PSNR} = 10 \log \left( \frac{255^2}{\| u_0 - u \|^2} \right) \text{dB}$$

(31)

where $u_0$ and $u$ are the original image and the inpainted image, respectively.

We compare the proposed algorithm (TDWI) with existing wavelet inpainting algorithms in [7, 10, 13] (abbreviated as TVWI, $p$-LaplaceWI, and NLTWI, respectively). $\Delta_t$, $N$, and $\sigma$ in TDWI are used to obtain the diffusion matrix. These parameters have a limited impact on the wavelet inpainting results. We use $\Delta_t = 0.01$, $N = 3$ and $\sigma = 0.1$ since empirically these choices give good results. According to [13], we set $\mu = 0.15$, $\lambda = 1.0$ for TVWI, and $\mu = 0.01$ for NLTWI. By default, parameter $\eta$ used in proximal iteration is set to $\eta = 1$. The wavelet decomposition level is 3.
4.1 The comparison of wavelet inpainting (WI) results

We compare the WI results with the same iteration step for all the four WI models in various loss cases. We set $N_{outer} = 50$ and $N_{inner} = 10$ for all the models.

1) Block loss

Fig. 3 shows the “boat” image with block loss and its restored results. In JPEG2000, the image is transformed into wavelet subbands and these wavelet coefficients are divided into codeblocks. During the transmission process, the loss or corruption of some codeblocks could affect the whole image quality. Fig. 3 (a) is the intact original image; Fig. 3 (b) is the block loss mask in the wavelet domain, black squares indicate each level of wavelet decomposition with top-left corner storing the coarsest low-low subband, which is referred to as “LL”, and the bottom-right storing the corresponding high-high ones. Fig. 3 (c) is the received image with the block loss. We can see that the loss in the LL subband causes the severest degradation and results in the black square stains in the image. The loss of other high frequencies in the coarsest subband creates Gibbs artifacts or other blur effects, while the loss of the rest frequency subbands do not greatly affect the image quality. From Figs. 3 (d) ∼ 3 (g), all these wavelet inpainting methods can fill in the missing information properly, but there is some residual blurry ghosting in black square regions in the TVWI, $p$-Laplace WI and NLTVWI reconstructed images, while TDWI can reconstruct perfectly the edges and corners and obtain better visual quality. For the zoomed sub-images, there emerge staircase artifacts in the flat region for TVWI, there exists blurring effect for $p$-Laplace WI, and there is an incompatible region in the black square. The proposed TDWI achieves a higher PSNR and a better visual quality than the other three methods.

2) Random loss

Fig. 4 shows the “Mickey” image with random loss and its inpainted images. In Fig. 4 (b), black squares indicate each level of wavelet decomposition (ditto). Because of the loss in LL frequency, the received image is of quite poor quality and irregular chunky stains emerge in Fig. 4 (c). From the subjective respect, TDWI restores the missing homogeneous information more properly and recovers better image structures simultaneously, including edges and corners. From the objective quantification respect, TDWI has an even higher PSNR than TVWI, $p$-Laplace WI and NLTVWI. In Fig. 5, we plot PSNR vs. the retained ratio for the case of random loss in the whole wavelet domain on the “Mickey” image. It shows that all the models can substantially improve image qualities. However, TDWI outperforms the other three and gives a higher PSNR at ratios from 10% to 90%. The proposed model can control the geometric regularity better in the pixel domain.

3) Noisy case

To further compare the models, we test the case when there is Gaussian noise in the random loss data. In Fig. 6, we introduce the Gaussian noise with standard deviation to a synthetic image. We can see that compared with the other three, TDWI recovers better not only the inner area but also the edges and corners. Besides, it gives a better denoising result due to its robustness to noise. In Fig. 7, we plot PSNR vs. the retained coefficient ratio for the case of random loss with noise on the synthetic image. TDWI outperforms the other three and gives a higher PSNR at ratios from 10% to 90%, since the regularization in the proposed model has better robustness to noise.

Fig. 3 The “boat” image with block coefficients lost: (a) Original image $512 \times 512$; (b) Block loss mask; (c) Received damaged image (PSNR = 16.2098); (d) Restored image by TVWI (PSNR = 25.7597); (e) Restored image by $p$-Laplace WI (PSNR = 27.7812); (f) Restored image by NLTVWI (PSNR = 29.9432); (g) Restored image by TDWI (PSNR = 31.8608); (h) ∼ (k) are the zoomed sub-images for (d) ∼ (g), respectively.
Fig. 4 The “Mickey” image with 50% wavelet coefficients randomly lost. (a) Original image $256 \times 256$; (b) Random loss mask; (c) Received damaged image (PSNR = 9.4932); (d) Restored image by TVWI (PSNR = 19.7356); (e) Restored image by $p$-LaplaceWI (PSNR = 20.9182); (f) Restored image by NLTVWI (PSNR = 22.5846); (g) Restored image by TDWI (PSNR = 24.4451); (h) $\sim$ (k) are the zoomed sub-images for (d) $\sim$ (g), respectively.

Fig. 5 PSNR vs. the retained ratio (on “Mickey” image without noise) (TDWI performs more effectively than the other three and gives a higher PSNR at each retained ratio.)

4.2 Comparison of execution time

We compare the execution time for all the four WI models with the same number of iteration steps. Table 1 shows the execution time with $N_{outer} = 50$ and $N_{inner} = 10$ in the cases of Figs. 3, 4, and 6.

<table>
<thead>
<tr>
<th>Time (s)</th>
<th>TVWI</th>
<th>p-LaplaceWI</th>
<th>NLTVWI</th>
<th>TDWI</th>
</tr>
</thead>
<tbody>
<tr>
<td>Boat</td>
<td>107.1</td>
<td>112.2</td>
<td>893.7</td>
<td>121.9</td>
</tr>
<tr>
<td>Mickey</td>
<td>124.2</td>
<td>130.2</td>
<td>913.6</td>
<td>142.1</td>
</tr>
<tr>
<td>synthetic</td>
<td>131.1</td>
<td>138.5</td>
<td>925.2</td>
<td>150.8</td>
</tr>
</tbody>
</table>

Since the computation of the diffusion matrix is time-consuming, the execution time of TDWI is slightly longer than the ones of TVWI and $p$-LaplaceWI. However, since the computation of the nonlocal weights in NLTVWI is very expensive, TDWI is much faster than NLTVWI.

In addition, we compare the convergence speeds of the four models in the random loss case of Fig. 4.

We set a fixed number of the inner iteration steps as $N_{inner} = 10$. Since the outer iteration performs the diffusion process, we test the PSNR of each WI model with $N_{outer}$ changing.

Fig. 8 shows the relationship between PSNR and $N_{outer}$. In the figure, TDWI converges to a stable state with much fewer iteration steps than the other three models. The tensor diffusion exploits more structure information to perform the structure-adaptive anisotropic regularization in the pixel domain. The proposed TDWI model adjusts adaptively the diffusion amount and diffusion direction according to the structure feature. Therefore, TDWI can give a better convergence speed.

Besides, Fig. 8 shows that all the WI models reach the steady state when $N_{outer} = 50$. Therefore, it is reasonable and fair that we compare the WI results by setting $N_{outer} = 50$ and $N_{inner} = 10$ for all the four models in Subsection 4.1.

5 Conclusion

We proposed a new and effective wavelet inpainting model based on tensor diffusion which combines the structure-adaptive anisotropic regularization with wavelet representation. According to its corresponding Euler-Lagrange equation, we analyzed the performance of the newly adopted regularization in local coordinates. We discussed different loss scenarios on different types of images,
with or without noise. The numerical examples have shown that the proposed model is very effective in filling in missing wavelet information and removing noise. The subjective quality and objective evaluation of the inpainted results are both improved greatly. Since the tensor diffusion exploits more structure information, the proposed model can reach a steady state with much fewer iteration steps.

Appendix A  Deducing the Euler-Lagrange equation of the proposed model

Let

$$\beta_I = \{ \beta_{j,k} | (j,k) \in I \} \quad (A1)$$

Some energy functional is of the type

$$F (u(\beta, x)) = \frac{1}{2} \int_{\Omega} \text{tr} (\Psi (J)) \, dx + \sum_{j,k} \frac{\lambda_{j,k}}{2} (\beta_{j,k} - \alpha_{j,k})^2 \quad (A2)$$

It follows that

$$\frac{\partial F (u(\beta, x))}{\partial \beta_{j,k}} =$$

$$\frac{1}{2} \int_{\Omega} \text{tr} (\Psi' (J)) \frac{\partial J}{\partial \beta_{j,k}} \, dx + \sum_{j,k} \lambda_{j,k} (\beta_{j,k} - \alpha_{j,k}) \quad (A3)$$

Its first term is

$$\frac{1}{2} \int_{\Omega} \text{tr} (\Psi' (J)) \frac{\partial J}{\partial \beta_{j,k}} \, dx =$$

$$\frac{1}{2} \int_{\Omega} \text{tr} (\Psi' (J)) \left[ \frac{\partial \nabla u(\beta, x)}{\partial \beta_{j,k}} (\nabla u(\beta, x))^T \right] \, dx +$$

$$\frac{1}{2} \int_{\Omega} \text{tr} (\Psi' (J)) \left[ \frac{\partial \nabla u(\beta, x)}{\partial \beta_{j,k}} (\nabla u(\beta, x))^T \right] \, dx +$$

$$\frac{1}{2} \int_{\Omega} \text{tr} (\Psi' (J)) \left[ \nabla u(\beta, x) \frac{\partial^2 u(\beta, x)}{\partial \beta_{j,k}} \right] \, dx =$$

$$\frac{1}{2} \int_{\Omega} \text{tr} (\Psi' (J)) \left[ \nabla u(\beta, x) \frac{\partial^2 u(\beta, x)}{\partial \beta_{j,k}} \right] \, dx$$
We know that $\Psi' (J) = \sum_i \Psi' (\mu_i) v_i v_i^T$, therefore,

$$\frac{1}{2} \int_{\Omega} \text{tr} \left( \Psi' (J) \right) \frac{\partial J}{\partial \beta_{j,k}} \, dx =$$

$$\frac{1}{2} \int_{\Omega} \sum_i \Psi' (\mu_i) v_i v_i^T \times$$

$$\left[ \nabla \psi_{j,k} (\nabla u (\beta, x)) + \nabla u (\beta, x) (\nabla \psi_{j,k})^T \right] \, dx =$$

$$\frac{1}{2} \int_{\Omega} \sum_i \Psi' (\mu_i) \left( v_i^T \nabla \psi_{j,k} \right) \left( v_i (\nabla u (\beta, x))^T + \left( v_i^T \nabla u (\beta, x), \nabla \psi_{j,k} \right) \right) \, dx$$

(A5)

For column vectors $a$ and $b$, $\text{tr} (ab^T) = a^T b$ and $\text{tr} (a^T) = \text{tr} (a)$, therefore,

$$\frac{1}{2} \int_{\Omega} \text{tr} \left( \Psi' (J) \right) \frac{\partial J}{\partial \beta_{j,k}} \, dx =$$

$$\int_{\Omega} \sum_i \Psi' (\mu_i) \left[ \left( \nabla u (\beta, x) v_i \right) v_i^T \nabla \psi_{j,k} \right] \, dx =$$

$$\int_{\Omega} \left( \nabla u (\beta, x) \right)^T \left( \sum_i \Psi' (\mu_i) v_i v_i^T \right) \nabla \psi_{j,k} \, dx =$$

$$\int_{\Omega} \left( \nabla u (\beta, x) \right)^T \Psi' (J) \nabla \psi_{j,k} \, dx =$$

$$\int_{\Omega} \left( \nabla u (\beta, x) \right)^T g (J) \nabla \psi_{j,k} \, dx$$

(A6)

Since $g (J)$ is a symmetric matrix, we can obtain that

$$\frac{1}{2} \int_{\Omega} \text{tr} \left( \Psi' (J) \right) \frac{\partial J}{\partial \beta_{j,k}} \, dx = \int_{\Omega} (g (J) \nabla u (\beta, x))^T \nabla \psi_{j,k} \, dx$$

(A7)

We assume that the mother wavelet (here we use Daubechies $7 \sim 9$) is compactly supported and at least Lipschitz continuous, therefore we can yield through integration-by-parts that

$$\frac{1}{2} \int_{\Omega} \text{tr} \left( \Psi' (J) \right) \frac{\partial J}{\partial \beta_{j,k}} \, dx =$$

$$- \int_{\Omega} \nabla \cdot (g (J) \nabla u (\beta, x))^T \nabla \psi_{j,k} \, dx$$

(A8)

For a vector $a$, $\nabla \cdot (a^T) = \nabla \cdot (a)$, thus

$$\frac{1}{2} \int_{\Omega} \text{tr} \left( \Psi' (J) \right) \frac{\partial J}{\partial \beta_{j,k}} \, dx =$$

$$- \int_{\Omega} \nabla \cdot (g (J) \nabla u (\beta, x)) \psi_{j,k} \, dx =$$

$$- \left( \nabla \cdot (g (J) \nabla u (\beta, x)), \psi_{j,k} \right)$$

(A9)

The Euler-Lagrange equation of the proposed model is

$$- \left( \nabla \cdot (g (J) \nabla u (\beta, x)), \psi_{j,k} \right) + \sum_{j,k} \lambda_{j,k} (\beta_{j,k} - \alpha_{j,k}) = 0$$

(A10)

References


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