Stability Analysis of Continuous-time Iterative Learning Control Systems with Multiple State Delays

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Abstract This paper presents a stability analysis of the iterative learning control (ILC) problem for continuous-time systems with multiple state delays, especially when system parameters are subject to polytopic-type uncertainties. Using the two-dimensional (2-D) analysis approach to ILC, the continuous-discrete Roesser’s type linear systems are employed to describe the entire learning dynamics of time-delay systems (TDS) with the development of an expanding operator. Based on such Roesser systems, the 2-D system theory is first used to develop a necessary and sufficient condition for the asymptotic stability of ILC, and then the robust $H_\infty$ control theory is combined to provide a sufficient condition in terms of linear matrix inequalities (LMIs) for the monotonic convergence of ILC. It is shown that learning gains can be determined by solving LMIs, which ensure the control input error converges monotonically to zero as a function of iteration. Simulation results show that a robust asymptotically stable ILC scheme can become robustly monotonically convergent by adding the P-type learning gains that satisfy a set of LMIs, which can also improve the convergence rate greatly.

Key words Iterative learning control (ILC), time-delay systems, monotonic convergence, 2-D system theory, robust $H_\infty$ control theory, linear matrix inequality (LMI)

Iterative learning control (ILC) is an effective technique for such systems that operate repetitively over a finite time interval. The key feature of ILC is to improve the control input law iteratively to achieve a perfect tracking by feeding back the control signals and the associated tracking errors from previous trials. During the iterative learning process, the reasonable transient behavior is desirable to practical applications rather than the asymptotic stability. If an ILC algorithm converges monotonically, good transients can be obtained. For delay-free systems, the system matrices are of appropriate dimensions, and each delay takes an integral multiple of the fixed delay time. Therefore, system (1) is the state, which is the output. The state $x(t, k)$ is the control input error converges monotonically in the sense of the $L_2$-norm. It is worth noting that all conditions are derived from using 2-D Roesser systems. But, compared with the results of ILC based on Roesser systems, the result of this paper provides new insights into stability analysis of the established 2-D error systems, into which the robust $H_\infty$ control theory is incorporated. In addition, an expanding operator is introduced based on the delay operator, which replaces time delays in the proofs of ILC stability. Hence, time delays do not play a significant role any longer, while in contrast, the ratio of the learning time length to the unit delay plays a crucial role in establishing the 2-D error systems and implementing LMIs with the Matlab toolbox, as shown by numerical simulation.

1 ILC system description

1.1 System description

Consider the following continuous-time system with multiple state delays, which is modeled in the 2-D form of

$\frac{\partial x(t, k)}{\partial t} = \sum_{\tau=0}^{L} A_i x(t-\tau, k) + B u(t, k)$

$y(t, k) = C x(t, k)$

where $t \in [0, T]$ is the continuous-time index, $k \in Z_+$ is the discrete-iteration index, $x(t, k) \in R^n$ is the state, $u(t, k) \in R^p$ is the control input, and $y(t, k) \in R^q$ is the output. The system matrices $A_i, i \in [0, l], B, C$ are of appropriate dimensions, and each delay takes an integral multiple of the fixed delay time $\tau$. In the following discussion, let us denote $\nabla$ as the pure delay operator, where $\nabla : \nu(t, k) \rightarrow \nu(t-\tau, k)$ and $\nu$ is a 2-D vector function defined on the time interval $[t-\tau, t]$ for all $k \in Z_+$, and for matrices $M_i, i \in [0, l]$, and let us denote $M(\nabla) = \sum_{i=0}^{l} M_i \nabla$ \cite{14}. Therefore, system (1) can be rewritten in the compact form of

$\frac{\partial x(t, k)}{\partial t} = A(\nabla) x(t, k) + B u(t, k)$

$y(t, k) = C x(t, k)$

The following assumptions on system (1) are imposed:

Assumption 1. It is assumed that $y(t)$ is a realizable desired output trajectory. That is, for any realizable trajectory and an appropriate initial function $\varphi(t)$, $t \in [-\tau, 0]$, there exists a unique control input $u(t)$ such that

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From (8), it is clear that if one takes \( \beta(t) = \beta_{\text{v}}(t) \), \( \delta\mathbf{x}(t, k) = \mathbf{x}(t) - \mathbf{x}(t, k) \), \( \delta\mathbf{u}(t, k) = \mathbf{u}(t) - \mathbf{u}(t, k) \), and \( \mathbf{e}(t, k) = \mathbf{y}(t) - \mathbf{y}(t, k) \) with \( \mathbf{x}(t) = \varphi_{\text{v}}(t), \), \( t \in [-l\tau, 0] \).

According to this assumption, let us define the state, control input, and tracking error vectors as \( \delta\mathbf{x}(t, k) = \mathbf{x}(t) - \mathbf{x}(t, k) \), \( \delta\mathbf{u}(t, k) = \mathbf{u}(t) - \mathbf{u}(t, k) \), and \( \mathbf{e}(t, k) = \mathbf{y}(t) - \mathbf{y}(t, k) \).

**Assumption 2.** Assume that the standard ILC initial condition is satisfied, i.e., \( \mathbf{x}(t, k) = \varphi_{\text{v}}(t) \) for \( t \in [-l\tau, 0] \) and \( k \in \mathbb{Z}^+ \).

A consequence of Assumption 2 is that the zero state error is achieved over the initial time interval \( [-l\tau, 0] \), i.e., \( \delta\mathbf{x}(t, k) = \mathbf{0} \) for all \( t \in [-l\tau, 0] \) and \( k \in \mathbb{Z}^+ \). If this assumption is not satisfied, then ILC may be thought of as a robust problem on initial issues\(^{[9,19]} \). That is, we assume that \( \delta\mathbf{x}(t, k) \neq \mathbf{0} \) for all \( t \in [-l\tau, 0] \), and the objective would be to find the control input \( \mathbf{u}(t, k) \) that can guarantee the stability of ILC robust against nonzero initial state errors. A possible way to solve the problem is to modify the desired trajectory at the initial stage by making an appropriate interpolation\(^{[19]} \).

**Assumption 3 (Parameter uncertainties assumptions).** It is assumed that the system matrices \( A_i, i \in \{0, l\} \) and \( B \) in (1) are known to lie within the uncertainty polytope \( \Omega \) where

\[
\Omega = \left\{ \left( A_0, \cdots, A_l, B \right) \mid \left( A_0, \cdots, A_l, B \right) = \sum_{p=1}^{N} \varsigma_p \left( A_{0p}, \cdots, A_{lp}, B_p \right) ; \varsigma_p \geq 0, \sum_{p=1}^{N} \varsigma_p = 1 \right\}
\]  

(4)

### 1.2 ILC scheme description

Our objective is to design an iterative scheme to generate the control input \( \mathbf{u}(t, k) \) such that the system output \( \mathbf{y}(t, k) \) converges to \( \mathbf{y}_d(t) \) and the control input \( \mathbf{u}(t, k) \) monotonically converges to \( \mathbf{u}_d(t) \) as \( k \) goes to infinity for all \( t \) within the time interval \([0, T]\).

To realize the above control objective, we consider a PD-type ILC implemented as follows:

1) Control input:

\[
\mathbf{u}(t, k) = \mathbf{a}(t, k) + \mathbf{\beta}(t, k)
\]  

(5)

2) P-type ILC law:

\[
\mathbf{a}(t, k) = \mathbf{a}(t, k - 1) + \mathbf{K}(\nabla)\delta\mathbf{x}(t, k - 1)
\]  

(6)

3) D-type ILC law:

\[
\mathbf{\beta}(t, k) = \mathbf{\beta}(t, k - 1) + \mathbf{\Gamma}\dot{\mathbf{e}}(t, k - 1)
\]  

(7)

where \( \mathbf{\beta}(t, k) = \mathbf{\beta}(t, k - 1) \) and \( \mathbf{\Gamma} \) is an \( r \times q \) matrix.

**Remark 1.** After some simple algebraic manipulations, we can rewrite the ILC scheme (5) ~ (7) in the form of

\[
\mathbf{u}(t, k) = \mathbf{u}(t, k - 1) + \mathbf{K}(\nabla)\delta\mathbf{x}(t, k - 1) + \mathbf{\Gamma}\dot{\mathbf{e}}(t, k - 1)
\]  

(8)

From (8), it is clear that if one takes \( \mathbf{K}(\nabla) = \mathbf{0} \) (the zero operator), the typical D-type ILC can be derived, which was considered for TDS\(^{[10]} \). However, only asymptotic stability of the ILC process is achieved, and the high-overshoot can be generated. In the following, we show that adding the P-type ILC law (6) into the control input (5) helps to achieve monotonic convergence of the ILC process (in the sense of the control input error converging monotonically as a function of iteration) to guarantee good learning transients.

### 2 2-D analysis approach to ILC

The ILC process of (1), (5) ~ (7) is essentially a 2-D system because of two independent indices: time \( t \) and iteration \( k \), which is usually formulated in Roesser systems\(^{[10,12]} \).

In this section, we develop a new route to achieve the 2-D ILC dynamics formulation of TDS with Roesser systems. Then, some lemmas on both stability and robust performance related to the Roesser systems are provided.

First, an expanding operator is developed based on the delay operator, which is defined by

\[
\omega(\nabla, j) : \mathbf{u}(t, k) \mapsto \mathbf{u}^j(t, k) = \begin{bmatrix} \mathbf{u}(t, k) \\ \nabla\mathbf{u}(t, k) \\ \vdots \\ \nabla^j\mathbf{u}(t, k) \end{bmatrix}
\]  

(9)

In particular, \( \omega(\nabla, 0) \) is obviously an identity operator.

#### 2.1 2-D system representation

Iterating the ILC scheme (5) ~ (7) at iterations \( k \) and \( k + 1 \), we can obtain

\[
\delta\mathbf{u}(t, k + 1) = \delta\mathbf{u}(t, k) - \mathbf{K}(\nabla)\delta\mathbf{x}(t, k) - \mathbf{\Gamma}\dot{\mathbf{e}}(t, k)
\]  

(10)

Subtracting (2) from (3), we have

\[
\frac{\partial\delta\mathbf{x}(t, k)}{\partial t} = \mathbf{A}(\nabla)\delta\mathbf{x}(t, k) + \mathbf{B}\delta\mathbf{u}(t, k)
\]  

(11)

Substituting (11) into (9), we thus get

\[
\delta\mathbf{u}(t, k + 1) = -\left[ \mathbf{K}(\nabla) + \mathbf{\Gamma}\mathbf{A}(\nabla) \right] \delta\mathbf{x}(t, k) + \left( \mathbf{I} - \mathbf{\Gamma}\mathbf{C}\mathbf{B} \right)\delta\mathbf{u}(t, k)
\]  

(12)

where \( \mathbf{I} \) is an identity matrix of appropriate orders. To deal with the delay operator \( \nabla \) in (10) and (12), we introduce an analysis strategy that considers the time interval \([0, T]\) in an ordered way of a time interval with the length of the unit delay time \( \tau \), that is, let \([0, T] = [0, \tau] \cup [\tau, 2\tau] \cup \cdots \cup [l\tau, T] \) with \( l\tau < T \) or \([0, T] = [0, \tau] \cup [\tau, 2\tau] \cup \cdots \cup [(l - 1)\tau, T] \) with \( l\tau = T \), where \( l\tau = \text{int}(T/\tau) \) and \( \text{int}() \) represents the integer part of value. Without any loss of generality, only the latter partition of the time interval \([0, T]\) is considered\(^1 \), and for the sake of the following analysis, we formulate the considered interval in a unified form of

\[
[0, T] \cup [(T, (l + 1)\tau)] = \bigcup_{j=0}^{l\tau} [j\tau, (j + 1)\tau]
\]  

(13)

\(^1\)Whatever case is discussed, we will technically extend the definitions of \( \delta\mathbf{x}(t, k) \) and \( \delta\mathbf{u}(t, k) \) to the time interval \([T, (l + 1)\tau] \), since \((T, (l + 1)\tau)\) does not work for the ILC stability analysis. In this sense, the considered time interval for both cases will end up with the same form as shown in (13).
Next, consider partitions \([j\tau, (j+1)\tau), j \in [0, t]\) separately. Define two expanding vectors as \(\delta \vec{x}(t, k) = \omega(\nabla, j) \delta \vec{u}(t, k)\) and \(\delta \vec{u}(t, k) = \omega(\nabla, j) \delta \vec{u}(t, k)\), and therefore in a compact form, it follows from using (10) and (12) that

\[
\Sigma^j : \begin{bmatrix} A_0 & A_1 & \cdots & A_{j-1} & A_j \\ 0 & A_0 & \cdots & \cdots & \cdots \\ \vdots & \vdots & \ddots & \cdots & \vdots \\ 0 & 0 & \cdots & A_0 & A_1 \\ 0 & 0 & \cdots & 0 & A_0 \end{bmatrix}, \quad j \leq l
\]

and

\[
A'_j = \begin{bmatrix} A_0 & A_e & \cdots & A_l & A_t \\ 0 & A_0 & \cdots & \cdots & \cdots \\ \vdots & \vdots & \ddots & \cdots & \vdots \\ 0 & 0 & \cdots & A_0 & A_e \\ 0 & 0 & \cdots & 0 & A_0 \end{bmatrix}, \quad j > l
\]

where \(A', B', C', I', K'\) and \(\Gamma'\) are defined in the same way as \(A, B, C, I, K, \Gamma\) for all \(j < l\) and \(j_o \in \mathbb{Z}_+\). \(\Sigma^j\) is degenerated to \(\Sigma^{j_o}\) in essence. Hence, the initial time of \(\Sigma^j\) indicates \(t = 0\) in the following discussion where no confusion arises. In particular, \(\Sigma^\infty\) can be used to clearly describe the whole learning dynamics of the ILC scheme \((5) \sim (7)\) over the time interval \([0, T]\).

### 2.2 Some lemmas

To derive the stability of 2-D Roesser systems as the only discrete index goes to infinity, the following lemma is useful and adopted from the literature.

**Lemma 2** (2-D system theory\(^{[16]}\)). Consider a 2-D continuous-discrete linear system described in the Roesser’s state-space model of

\[
\begin{bmatrix} \frac{\partial \psi(t)}{\partial t} \\ \zeta(t, k+1) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} \psi(t, k) \\ \zeta(t, k) \end{bmatrix} \quad (15)
\]

where \(\psi(t, k) \in \mathbb{R}^n\) and \(\zeta(t, k) \in \mathbb{R}^r\) are the horizontal state and the vertical state, respectively. If \(A_{11}, A_{12}, A_{21}, A_{22}\) are constant matrices of appropriate dimensions, and boundary conditions for system (15) satisfy that \(\psi(0, k) = 0\) holds for all \(k \in \mathbb{Z}_+\) and \(\zeta(t, 0)\) is finite for all \(t \in [0, T]\), then \(\lim_{k \to \infty} \psi(t, k, T^T) \zeta(t, k)^T = 0\) is satisfied for all \(t \in [0, T]\) if and only if (iff) the matrix \(A_{22}\) is stable, that is, the spectral radius fulfills \(\rho(A_{22}) < 1\).

For any fixed iteration \(k\), the 2-D system \((14)\) can also be taken as a one-dimensional \((1-D)\) system. That is, \(\delta \vec{x}(t, k)\) is the system state, \(\delta \vec{u}(t, k+1)\) is the exogenous disturbance signal, and \(\delta \vec{u}(t, k+1)\) is the objective function signal to be attenuated. Hence, to derive the (robust) \(H_{\infty}\) performance of such 1-D systems, the following lemmas are developed or directly adopted from the literature.

**Lemma 2** (Bounded real lemma (BRL)\(^{[18]}\)). Consider the following system

\[
\begin{align*}
\dot{\vec{z}}(t) &= \vec{A}(t) \vec{z}(t) + \vec{B} \vec{w}(t) \\
\vec{z}(t) &= \vec{C}(t) \vec{z}(t) + \vec{D} \vec{w}(t), \quad \vec{z}(0) = 0
\end{align*}
\quad (16)
\]

where \(\vec{z}(t) \in \mathbb{R}^n\) is the system state, \(\vec{w}(t) \in \mathbb{L}_2^2[0, \infty)\) is the exogenous disturbance signal, \(\vec{x}(t) \in \mathbb{R}^r\) is the objective function signal to be attenuated, and \(\vec{A}, \vec{B}, \vec{C}\) and \(\vec{D}\) are constant matrices of appropriate dimensions. Then, system (16) is asymptotically stable, and its \(H_{\infty}\)-norm is less than a prescribed scalar \(\gamma > 0\) iff there exists a positive definite symmetric matrix \(P > 0\) that satisfies

\[
\begin{bmatrix} \vec{A}^T \vec{P} + \vec{P} \vec{A} & \vec{P} \vec{B} & \vec{C}^T \\ \vec{B}^T \vec{P} & -\gamma I & \vec{D}^T \\ \vec{C} & \vec{D} & -\gamma I \end{bmatrix} < 0
\quad (17)
\]
Lemma 3 (Improved bounded real lemma (IBRL)\textsuperscript{[18]}). System (16) is asymptotically stable and its $H_{\infty}$-norm is less than a prescribed scalar $\gamma > 0$ if there exist a positive definite symmetric matrix $Q$, a matrix $Z$ and a sufficiently small positive scalar $\epsilon > 0$ that satisfy

$$\begin{bmatrix}
Q - Z - Z^T & Z^T + \epsilon Z^T A^T & 0 & Z^T C^T \\
Z + \epsilon A Z & -Q & B & 0 \\
0 & B^T & -\epsilon^{-1} I & -\epsilon^{-1} D^T \\
C Z & 0 & 0 & -\epsilon^{-1} D - \epsilon^{-1} I
\end{bmatrix} < 0 \tag{18}
$$

Lemma 4. If the system matrices of (16) are known to lie within the following uncertainty polytope

$$\hat{\Omega} = \left\{ (A, B, C, D) \mid \begin{array}{c}
\sum_{p=1}^{N} \eta_p (A_p, B_p, C_p, D_p); \eta_p \geq 0, \sum_{p=1}^{N} \eta_p = 1
\end{array} \right\} \tag{19}
$$

then system (16) over $\hat{\Omega}$ is robustly asymptotically stable and its robust $H_{\infty}$-norm is less than a prescribed scalar $\gamma > 0$ if there exist positive definite symmetric matrices $Q_p$, $p \in [1, N]$, a matrix $Z$, and a sufficiently small positive scalar $\epsilon > 0$ that satisfy

$$\begin{bmatrix}
Q_p - Z - Z^T & Z^T + \epsilon Z^T A_p^T & 0 & Z^T C_p^T \\
Z + \epsilon A_p Z & -Q_p & B_p & 0 \\
0 & B_p^T & -\epsilon^{-1} I & -\epsilon^{-1} D^T \\
C_p Z & 0 & 0 & -\epsilon^{-1} D - \epsilon^{-1} I
\end{bmatrix} < 0 \quad p \in [1, N] \tag{20}
$$

Proof. If the set of LMIs in (20) holds, then we denote $Q = \sum_{p=1}^{N} \eta_p Q_p$. We can easily show that $Q$ is a positive definite symmetric matrix and can be used as matrix $Q$ of LMI (18), which is required by the system matrices over $\hat{\Omega}$. Using Lemma 3, this proof is immediate. \qed

Lemma 5 (Schur complement\textsuperscript{[18]}). Given a symmetric matrix $S$ with the form $S = S_1 \oplus S_2$, where $S_1 \in \mathbb{R}^{r \times r}$, $S_2 \in \mathbb{R}^{(n-r) \times (n-r)}$, then $S < 0$ if $S_1 < 0$ and $S_2 - S_1 S_1^T S_2 < 0$, or, $S_2 < 0$ and $S_1 - S_2 S_2^T S_1 < 0$.

Besides the above results of systems theory, the following property of operator $\omega(\nabla, j)$, $\forall j \in \mathbb{Z}_+$ is also useful.

Lemma 6. Given a 2-D function $\nu(t, k)$, if for any $k \in \mathbb{Z}_+$, it belongs to the space of square integrable vector functions over $[0, \infty)$ and $\nu(t, k) = 0$ holds for $t \leq 0$, then

$$\|\nu(t, k)\|_2 = \sqrt{J} \|\nu(t, k)\|_2 \tag{21}
$$

where $\nu(t, k) = \omega(\nabla, j) \nu(t, k)$, $\forall j \in \mathbb{Z}_+$.

Proof. From the definition of $\omega(\nabla, j)$, $\forall j \in \mathbb{Z}_+$, we get

$$\|\nu(t, k)\|_2^2 = \int_0^{\infty} (\nu(t, k))^T (\nu(t, k)) \, dt = \int_0^{\infty} \sum_{m=0}^{j} (\nabla^m (\nu(t, k)))^T (\nabla^m (\nu(t, k))) \, dt \tag{22}
$$

where $\nu(t, k) = 0$ for all $t \leq 0$ and $k \in \mathbb{Z}_+$. From (22), the proof of (21) is immediate. \qed

3 Stability conditions of ILC

In this section, we consider the asymptotic stability and monotonic convergence of ILC separately.

3.1 Asymptotic stability

With Lemma 1 and based on the 2-D system (14), one can state the following result.

Theorem 1. Let system (1) satisfy Assumptions 1 and 2, and ILC scheme (5) \& (7) be applied. The state error $\delta z(t, k)$, the control input error $\delta u(t, k)$, and the tracking error $\delta e(t, k)$ converge to zero as $k \rightarrow \infty$ for all $t$ within the time interval $[0, T]$ if the learning gain matrix $\Gamma$ can be designed such that the matrix $I - \Gamma C B$ is stable.

Proof. Note that the 2-D system $\Sigma^T$ in (14) can be considered over the time interval $[0, T]$ (see Remark 4), which satisfies: $\delta x^T (0, k) = 0$ for all $k \in \mathbb{Z}_+$, and finite $\delta u^T (t, 0)$ for all $t \in [0, T]$. Thus, it follows by using Lemma 1 that

$$\lim_{k \rightarrow \infty} [\delta x^T (t, k)^T \delta u^T (t, k)] = 0
$$

is satisfied for all $t \in [0, T]$ and therefore (see Remark 3)

$$\lim_{k \rightarrow \infty} [\delta x^T (t, k) \delta u^T (t, k)] = 0
$$

is satisfied for all $t \in [0, T]$ iff the matrix $I^T - I^T C^T B^T$ is stable. Since $e(t, k) = C \delta x(t, k)$ and $I^T - I^T C^T B^T$ are stable if $I - \Gamma C B$ is stable, the proof is immediate. \qed

Corollary 1. Let system (1) satisfy Assumptions 1 \& 3, and ILC scheme (5) \& (7) be applied. The state error $\delta z(t, k)$, the control input error $\delta u(t, k)$, and the tracking error $e(t, k)$ converge to zero as $k \rightarrow \infty$ for all $t$ within the time interval $[0, T]$ if the learning gain matrix $\Gamma$ can be designed such that $\max_{\epsilon \in [1, N]} \|I - \Gamma C B\| < 1$, where $\| \cdot \|$ can be used by picking any kind of matrix norm such as the $\infty$-norm $\| \cdot \|_\infty$.

Proof. Using the fact that $B = \sum_{p=1}^{N} \eta_p B_p$, we have

$$\|I - \Gamma C B\| = \|I - \Gamma C \sum_{p=1}^{N} \eta_p B_p\| = \|\sum_{p=1}^{N} \eta_p (I - \Gamma C B_p)\| \leq \sum_{p=1}^{N} \eta_p \|I - \Gamma C B_p\| < \sum_{p=1}^{N} \eta_p = 1
$$

Hence, $\rho(I - \Gamma C B) < \|I - \Gamma C B\| < 1$, namely, the matrix $I - \Gamma C B$ is stable. According to Theorem 1, this proof is immediately completed. \qed

Remark 5. Theorem 1 implies that the operator $K(\nabla)$ has no effect on conditions for the asymptotic stability of ILC, and such conditions are independent of the system
dynamics appearing in operator $\mathbf{A}(\nabla)$. Note that the stability of $I - \Gamma CB$ implies that the matrix $CB$ has full-column rank. Thus, if gain matrices of the ILC scheme (5) $\sim$ (7) are selected as $K_i = -\Gamma CA_i$, $i \in [0, l]$, and $\Gamma = [(CB)^T CB]^{-1} (CB)^T$, then the tracking error can be driven to zero over the whole time interval $[0, T]$ only after one learning iteration. This type of ILC design has been considered by many authors in the literature $[10, 13, 16]$ to improve the system performance. To our knowledge, there is no reference in the literature stating that it is straightforward to derive this type of fast ILC by adding the pure error terms in the typical D-type updating law.

In Corollary 1, a sufficient condition is derived for robust asymptotic stability of ILC when the TDS in (1) are subject to polytopic-type uncertainties. It is shown that checking the vertex impulse response matrices of a polytope plant is sufficient to determine the stability properties of the polytope ILC system. To our knowledge, such a robust ILC problem has not been studied by the 2-D analysis approach.

### 3.2 Monotonic convergence

Next, the (robust) monotonic convergence of time-delay ILC systems will be developed by considering Lemmas 2 $\sim$ 6.

**Theorem 2.** Let system (2) satisfy Assumptions 1 and 2, and ILC scheme (5) $(\sim)$ be applied. If there exist a positive definite symmetric matrix $P > 0$ and matrices $X$ and $Y$ such that the following LMI holds with $j = l_T - 1$

\[
\begin{bmatrix}
A^T P + PA^T & PB^T & X^T + A^T C^T j^T Y^T \\
B^T P & -I^T & Y^T + B^T C^T j^T Y^T \\
X + YC^T & I^T + YC^T B^T & -I^T
\end{bmatrix} < 0
\]  

then $\|\delta x(t, k)\|_{[2, 0], 1}$, $\|\delta u(t, k)\|_{[2, 0], 1}$, and $\|e(t, k)\|_{[2, 0], T}$ are bounded for all $k \in \mathbb{Z}_+$, and $\lim_{k \to \infty} \|\delta x(t, k)\|_{[2, 0], T} = 0$, $\lim_{k \to \infty} \|e(t, k)\|_{[2, 0], T} = 0$, and $\lim_{k \to \infty} \|\delta u(t, k)\|_{[2, 0], T} = 0$ (monotonic convergence in the sense of the $L_2$-norm). If LMI (23) holds, then learning gain matrices $K_i$, $i \in [0, l]$ and $\Gamma$ are computed by

\[
K_i = -[I \quad 0_{n \times (l_T - i)}] X \begin{bmatrix} 0_{n \times n} \quad I \quad 0_{n \times (l_T - (i + 1)n)} \end{bmatrix}^T \\
\Gamma = -[I \quad 0_{n \times (l_T - i)}] Y \begin{bmatrix} 0_{n \times n} \quad I \quad 0_{n \times (l_T - (i + 1)n)} \end{bmatrix}^T
\]  

**Proof.** If LMI (23) holds and learning gain matrices $K_i$, $i \in [0, l]$ and $\Gamma$ are used as in (24), it follows from using Lemma 2 that

\[
\|G'(s)\|_{\infty} < 1
\]  

where $G'(s)$ is given by

\[
G'(s) = \begin{bmatrix} A^T & B^T \\
-K^T - \Gamma C^T A^T & P - \Gamma C^T B^T \\
(K^T + \Gamma C^T A^T) (sI^T - A^T) -1 B^T + (P - \Gamma C^T B^T)
\end{bmatrix} < 0
\]  

As a consequence of (14), it follows that

\[
\delta U'(s, k + 1) = G'(s) \delta U'(s, k)
\]

where $\delta U'(s, k) = \mathcal{L} \left[ \delta u'(t, k) \right]$. Hence, using the fact that $\|\delta U'(s, k)\|_2 = \|\delta u'(t, k)\|_2$, we get

\[
\|\delta u'(t, k + 1)\|_2 \leq \|G'(s)\|_{\infty} \|\delta u'(t, k)\|_2
\]  

Since Lemma 6 results in $\|\delta u'(t, k)\|_2 = \sqrt{j + 1} \|\delta u(t, k)\|_2$, (28) leads to

\[
\|\delta u(t, k)\|_2 \leq \|G'(s)\|_{\infty} \|\delta u(t, k - 1)\|_2 \leq \left( \|G'(s)\|_{\infty} \right)^j \|\delta u(t, 0)\|_2
\]  

Using $j = l_T - 1$, system (14), and therefore $G'(s)$, can work over $[0, T]^3$ for the ILC system (1), $(\sim)$ (7) (see Remark 6). Note that at the first iteration, i.e., for $k = 0$, $\|\delta u(t, 0)\| = 0$, and thus $\|\delta u(t, 0)\|_{[2, 0], T} = \|\delta u(t, 0)\|_{[0, T]}$ is bounded. Consequently, over the finite time interval $[0, T]$, one can conclude from (29) that $\|\delta u(t, k)\|_{[2, 0], T}$ is bounded for all $k \in \mathbb{Z}_+$, and $\lim_{k \to \infty} \|\delta u(t, k)\|_{[2, 0], T} = 0$ (monotonic convergence in the sense of the $L_2$-norm), where (25) is used.

Also, we can derive from (14) that

\[
\|\delta x(t, k)\|_2 \leq \|\delta x(t, k)\|_{[2, 0], T} \leq \|G'(s)\|_{\infty} \|\delta u(t, k)\|_2 \leq \|G'(s)\|_{\infty} \|\delta u(t, k)\|_2 = \|G'(s)\|_{\infty} \|\delta u(t, k)\|_2
\]

Using Lemma 5, we know that if LMI (23) holds, then

\[
A^T P + B^T \Gamma = 0 \quad \Gamma \quad A^T = 0\]  

which together with the Lyapunov theorem ensures that the matrix $A^T$ is exponentially stable. This implies the boundedness of $\|G'(s)\|_{\infty}$ and $\|G'(s)\|_{\infty}$. From (30), it thus follows that $\|\delta x(t, k)\|_{[2, 0], T}$ and $\|e(t, k)\|_{[2, 0], T}$ are bounded for all $k \in \mathbb{Z}_+$, and $\lim_{k \to \infty} \|\delta x(t, k)\|_{[2, 0], T} = 0$, and $\lim_{k \to \infty} \|e(t, k)\|_{[2, 0], T} = 0$.

**Theorem 3.** Let system (2) satisfy Assumptions 1 and 2, and ILC scheme (5) $(\sim)$ be applied. If the learning gain $\Gamma$ can be designed such that for a sufficiently small scalar $\epsilon > 0$, there exist a positive definite symmetric matrix $Q > 0$ and matrices $X$ and $Y$ satisfying the following LMI with $j = l_T - 1$

\[
\begin{bmatrix} X + \epsilon A^T X & T \quad 0 \\
X + \epsilon A^T X & -Q \\
T & -C^T D^T \quad 0 \\
0 & -C^T D^T \quad -C^T D^T
\end{bmatrix} < 0
\]  

then $\|\delta x(t, k)\|_{[2, 0], T}$, $\|\delta u(t, k)\|_{[2, 0], T}$, and $\|e(t, k)\|_{[2, 0], T}$ are bounded for all $k \in \mathbb{Z}_+$, and $\lim_{k \to \infty} \|\delta x(t, k)\|_{[2, 0], T} = 0$, and $\lim_{k \to \infty} \|e(t, k)\|_{[2, 0], T} = 0$.

**Proof.** The proof is established by considering Lemma 3 and following the proof of Theorem 2. □
Now, using Theorem 3, one can state the following result related to the robust monotonic convergence of ILC against polytopic-type uncertainties.

**Corollary 2.** Let system (2) satisfy Assumptions 1 ~ 3, and ILC scheme (5) ~ (7) be applied. If the learning gain \( \Gamma \) can be designed such that for a sufficiently small scalar \( \epsilon > 0 \), there exist positive definite symmetric matrices \( Q_p \), \( F_p \in [1, N] \) and matrices \( X \) and \( Y \) satisfying the following LMIs with \( j = l_T - 1 \),
\[
\begin{bmatrix}
Q_0 - X - X^T & (X + A_J^T X)^T & 0 \\
X + A_J^T X & -Q_2 & B_J^T \\
0 & -\epsilon^{-1} D_{j}^T - \epsilon^{-1} D_{j}^T & 0
\end{bmatrix} < 0, \quad p \in [1, N]
\]
then \( \| \delta z(t, k) \|_2 \) and \( \| u(t, k) \|_2 \) are bounded for all \( k \in Z_+ \), and
\[
\lim_{t \to \infty} \| \delta z(t, k) \|_2 = 0, \quad \text{and} \quad \lim_{t \to \infty} \| u(t, k) \|_2 = 0.
\]

**Proof.** The proof is established by considering Lemma 4 and following the proof of Theorem 2. □

**Remark 6.** From Theorems 2 and 3, it is clear that the conditions for monotonic convergence are dependent on the system dynamics appearing in the operator \( \mathbf{A}(\nabla) \). This, together with Remark 5, implies that the difference between conditions for monotonic convergence and for asymptotic stability is large. Using Lemma 5 to LMI (33) hold, learning gain matrices \( K_i \), \( i \in [0, l] \) can also be computed by (32).

**Remark 7.** Corollary 2 implies that a stable ILC design of TDS with polytopic-type uncertainties can become monotonically convergent, and the only requirement is that the set of LMIs in (33) is satisfied. If this condition holds, the operator \( \mathbf{K}(\nabla) \) (33) is also computed. This results from that the D-analysis approach can convert the time-delay ILC systems into the traditional D-input-output systems of the control input errors between two sequential iterations. Consequently, the monotonic convergence property of ILC is transformed into the robust \( H_\infty \) system performance, based on which the IBRL [18] can be applied to deal with the polytopic-type uncertainties.

**Remark 8.** If one takes \( \tau = 0 \) in system (1), and following the same steps of the proofs of Theorem 1 and Corollary 1, one can prove that conditions for the (robust) asymptotic stability of ILC still work. It seems that the delays do not play a significant role in the stability analysis of ILC. Furthermore, it is an interesting fact that the implementations of LMIs (32), (31), and (33) depend not on the delay \( \tau \) but on \( l_T \), i.e., the ratio of the learning time length \( T \) to the delay \( \tau \). This results from that the expanding operator \( \mathbf{N}(\nabla, j) \) covers the ill-effect of the delay operator \( \nabla \) in the ILC analysis of the TDS in (1). In particular, Theorems 2 and 3 and Corollary 2 imply that if \( l_T \) keeps the same value, the same conditions for the monotonic convergence of ILC can be derived regardless of the values of \( T \) and \( \tau \).

**4. Simulation results**

In this simulation test, let us consider the TDS in (1) with \( l = 1 \) and matrices given by
\[
A_0 = \begin{bmatrix} 0 & 1 \\ -63 & -16 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 0 \\ 60 & -60 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 + g \end{bmatrix}
\]
and \( C = [0 \ 1] \), where \( g \) is an uncertain parameter known to reside in the following polytope:
\[
\Omega_g = \{ g | g = c_1 g_1 + c_2 g_2; g_1 = -0.5, g_2 = 0.5 \}\]
\( \chi_1 \geq 0, \chi_2 \geq 0, \chi_1 + \chi_2 = 1 \}

Let the system state be \( x(t) = [x_1(t) \ x_2(t)]^T \) and the desired trajectories be described by
\[
x_d(t) = [x_{d1}(t) \ x_{d2}(t)] = \begin{bmatrix} 4t^2 - 3t^3 \ 12t^2(1+t) \end{bmatrix}, \quad t \in [0, T]
\]
and \( x_d(t) = 0, \ t \in [-\tau, 0] \). For different pairs of the delay \( \tau \) and learning time \( T \), the following two cases are simulated:

**Case I.** \( \tau = 0.2, \ T = 1 \)

**Case II.** \( \tau = 0.3, \ T = 1.5 \)

where \( l_T = 5 \) holds in both cases. Asymptotic stability and monotonic convergence are considered separately.

**4.1 Asymptotic stability**

For the asymptotic stability, we consider the ILC scheme (5) ~ (7) without the \( P \)-type law, i.e., taking \( K(t) = 0 \) in (6) and with the \( D \)-type law (7) using \( \Gamma = 0.5 \). From Corollary 1, it follows that \( \max \{ \| I - \Gamma CB \|, \| I - \Gamma CB \| \} = 0.75 \). Hence, the ILC system is robustly asymptotically stable. Fig. 1 shows test results, where without any loss of generality we pick parameters \( c_1 = 0.6 \) and \( c_2 = 0.4 \) that are within the polytope uncertainty \( \Omega_g \). In Fig. 1, we describe simulation results of Case I in the upper three figures and those of Case II in the lower three figures. For each case, the \( L_2 \)-norms of state errors \( ||x(t, k) - x_d(t, k)||_2 \) and control input errors \( ||u(t, k) - u(t, k)||_2 \) are described in Fig. 1 (a) ~ (c), respectively. Obviously, the ILC system is robustly stable. However, the system is only asymptotically stable, and from each subfigure, we observe that even though the ILC system converges as iteration \( k \) increases, large transient growth is generated.

Now, if we test the LMI (25) with \( j = l_T - 1 \), \( X = 0 \), and \( Y = -diag(0.5, 0.5, 0.5, 0.5, 0.5) \), we can obviously see that this LMI is infeasible (in this case, the index \( t_{\min} = 9.71099 \) is nonnegative). According to Theorem 2, the monotonic convergence of ILC cannot be guaranteed to prevent large learning transient growth with such leaning gains.

**4.2 Monotonic convergence**

For the monotonic convergence, we use the same \( D \)-type learning gain \( \Gamma = 0.5 \) as used in the asymptotic stability tests. For the operator \( K(\nabla) \) in the \( P \)-type ILC law, it can be determined by solving the set of LMIs in (33) with \( j = 4 \) and \( \epsilon = 0.01 \), which is computed by (as given in (32))
\[
K(\nabla) = K_0 + K_1 \nabla, \quad K_0 = \begin{bmatrix} 31.5 \ 8 \end{bmatrix}, \quad K_1 = \begin{bmatrix} -30 \ 30 \end{bmatrix}
\]
The set of LMIs in (33) is feasible in this case, since it holds that \( t_{\min} = -0.4733 \). From Corollary 2, it is clear that the ILC system will be robustly monotonically convergent. Fig. 2 shows the test results of such determined ILC scheme (5) ~ (7) in both Cases I and II. As shown in this figure, the \( L_2 \)-norms of the state errors and control input error decay monotonically to zero as the learning trial increases. Moreover, the convergence rate is greatly improved, which can be clearly seen by comparing Fig. 1 with Fig. 2.
The simulation tests verify that the LMI approach can be used to achieve good learning transients of uncertain TDS by converting an asymptotically stable ILC into a monotonically convergent ILC, in which not the delay operator but the expanding operator plays a significant role.

5 Conclusions

In this paper, a PD-type ILC has been discussed for TDS subject to polytopic-type uncertainties. After the 2-D ILC error systems are derived, an LMI approach is introduced to deal with such 2-D systems that disclose a connection between the control input errors of two sequential iterations. Hence, this paper considers the asymptotic stability of ILC by the 2-D system theory and the monotonically convergent of ILC by the BRL (IBRL). In particular, sufficient LMI conditions are presented, which provide criterions to determine learning gains to achieve the robust monotonic convergence of ILC. These results have been verified through numerical simulation tests.

References


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