$H_\infty$ Output Feedback Control Design of Fuzzy Dynamic Systems via LMI

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Abstract This paper presents $H_\infty$ output feedback control design of complex nonlinear systems which can be represented by a fuzzy dynamic model. Based on a common Lyapunov function and a piecewise differentiable Lyapunov function respectively, two kinds of new $H_\infty$ output feedback fuzzy control design methods are developed. The $H_\infty$ output feedback controller can be obtained by solving a set of suitable linear matrix inequalities. An example is given to demonstrate the application of the proposed design methods.

Key words Fuzzy dynamic systems, $H_\infty$ output feedback control, LMI.

1 Introduction

Fuzzy logic control (FLC) based on the conception of fuzzy sets proposed in [1] has been an appealing control approach$^{[2\sim6]}$. During the last few years, a number of new methods have been proposed for the systematic analysis and design of fuzzy logic controllers based on the so-called fuzzy dynamic model, which consists of a family of local linear models smoothly connected through fuzzy membership functions$^{[7\sim19]}$.

$H_\infty$ control has been an attractive research topic during the last decade$^{[20\sim22]}$. So far a couple of papers$^{[10,15]}$ have discussed the $H_\infty$ feedback control for fuzzy dynamic systems. However, those papers dealt with a state feedback control design that requires all states of systems to be measured. In many cases, this requirement is too restrictive. Therefore, an $H_\infty$ output feedback control design for fuzzy dynamic systems is warranted. It should be noted that the state feedback control design methods can not be easily modified for the output feedback control design for fuzzy dynamic systems. Recently, there appeared a few results of output feedback control design for fuzzy dynamic systems$^{[17,19]}$. However the result in [17] is not constructive, many trials might be needed before an acceptable controller is found; the other discusses $H_2$ output feedback control design$^{[19]}$. In this paper, we will develop two new constructive $H_\infty$ output feedback control design methods for fuzzy dynamic systems by using LMI technique$^{[22]}$.

The rest of this paper is organized as follows. Section 2 introduces the fuzzy dynamic model. Based on a common Lyapunov function and a piecewise differentiable Lyapunov function, Section 3 and Section 4 present two new $H_\infty$ output feedback control design methods respectively. An example is given in Section 5. Finally, Section 6 concludes with some remarks.

Notation. The notation used is fairly standard. We denote by $z^Tz$ or $\|z\|^2_2$ the square norm of a vector $z \in \mathbb{R}^n$. The notation $L_2(0,T)$ will also be used for vector-valued functions, i.e. we say that $z : (0,T) \rightarrow \mathbb{R}^n$ is in $L_2(0,T)$ if $\int_0^T \|z(t)\|^2dt < \infty$.
2 Fuzzy Dynamic Model

For a nonlinear system, it is difficult, if not impossible, to design a satisfactory control in general. However, it is known that many nonlinear systems can be expressed as a set of linear systems in local operating regions. Therefore, in this paper, we will consider the following class of nonlinear systems, which can be represented by the socalled fuzzy dynamic model, that is,

\[
R^l: \text{IF } q_1 \text{ is } F^l_1 \text{ and } \cdots q_s \text{ is } F^l_s \text{ THEN } \dot{x}(t) = A_l x(t) + B_l u(t) + B_{wl} w(t), \\
\quad y(t) = C_l x(t), \\
\quad z(t) = C_{zl} x(t), \quad l = 1, 2, \cdots, m,
\]

where \( x(t) \in \mathbb{R}^n \) are the state variables, \( u(t) \in \mathbb{R} \) is the input variable, \( y(t) \in \mathbb{R} \) is the output variable, and \( w(t) \in \mathbb{R} \) is the disturbance variable which is deterministic signal that belong to \( L_2(0, \infty) \), \( z(t) \in \mathbb{R}^m \) are the controlled output variables, \( R^l \) denotes the \( l \)-th approximation inference rule, \( m \) is the number of approximation inference rules, \( F^l_j (j = 1, 2 \cdots s) \) are fuzzy sets, \( (A_l, B_l, B_{wl}, C_l, C_{zl}) \) represents the \( l \)-th local model of the nonlinear system, and \( q(t) := [q_1, q_2, \cdots, q_s] \) are some measurable system variables.

By using the fuzzy inference methods described in [7 ~ 19], that is using a center-average defuzzifier, product inference and singleton fuzzifier, the dynamic fuzzy model (1) can be expressed by the following global model

\[
\begin{align*}
\dot{x}(t) &= A(\mu(t))x(t) + B(\mu(t))u(t) + B_{wl}(\mu(t))w(t), \\
y(t) &= C(\mu(t))x(t), \\
z(t) &= C_z(\mu(t))x(t), \\
A(\mu(t)) &= \sum_{l=1}^{m} \mu_l A_l, \quad B(\mu(t)) = \sum_{l=1}^{m} \mu_l B_l, \quad B_{wl}(\mu(t)) = \sum_{l=1}^{m} \mu_l B_{wl}, \\
C(\mu(t)) - C_z(\mu(t)) &= \sum_{l=1}^{m} \mu_l C_l, \\
\mu(q(t)) := \mu(t) &= [\mu_1(t), \mu_2(t), \cdots, \mu_m(t)],
\end{align*}
\]

where \( \mu_l(q(t)) \) is the normalized membership function of the inferred fuzzy set \( F^l_i \) where

\[
F^l_i = \prod_{l=1}^{n} F^l_{i_l}
\]

and

\[
\sum_{l=1}^{m} \mu_l = 1.
\]

Define \( M \) as a set of membership functions satisfying (3).

This paper focuses on the \( H_\infty \) output feedback control design for fuzzy dynamic systems. The \( H_\infty \) performance is defined as follows.

**Definition 1.** Given a constant \( \gamma > 0 \), a closed loop system is stable with the \( H_\infty \) performance bound \( \gamma \) if the following conditions are satisfied:

1) the closed loop system is asymptotically stable,
2) with zero initial condition, the controlled output satisfies \( \|z\| \leq \gamma \|w\| \).
3 $H_{\infty}$ Output Feedback Control Design based on a Common Lyapunov Function

We will present one method based on a common Lyapunov function in this section. For the fuzzy system (2), a smooth output feedback controller can be chosen as follows

$$
\begin{align*}
\dot{x}_c &= \sum_{i=1}^{m} \mu_i A_{ci} x_c(t) + \sum_{i=1}^{m} \mu_i B_{ci} y(t), \\
u &= \sum_{i=1}^{m} \mu_i C_{ci} x_c(t).
\end{align*}
$$

(4)

With the output feedback controller (4), the closed loop fuzzy control system can be described as

$$
\begin{align*}
\dot{\hat{x}}(t) &= \tilde{A}(\mu)\hat{x}(t) + \tilde{B}(\mu)w(t), \\
z(t) &= \tilde{C}(\mu)\hat{x}(t),
\end{align*}
$$

(5)

$$
\tilde{A}(\mu) = \begin{bmatrix} 
\sum_{i=1}^{m} \mu_i A_i & \sum_{i=1}^{m} \mu_i B_i & \sum_{i=1}^{m} \mu_i C_i \\
\sum_{i=1}^{m} \mu_i B_{ci} & \sum_{i=1}^{m} \mu_i C_{ci} & \sum_{i=1}^{m} \mu_i A_{ci}
\end{bmatrix}, \quad
\tilde{B}(\mu) = \begin{bmatrix} \sum_{i=1}^{m} \mu_i B_{wi} \\
0
\end{bmatrix},
$$

$$
\tilde{C}(\mu) = \begin{bmatrix} \sum_{i=1}^{m} \mu_i C_{zi} & 0 \\
0 & x_c
\end{bmatrix}, \quad \hat{x} = \begin{bmatrix} x \\
x_c
\end{bmatrix}, \quad \forall \mu \in M.
$$

For the closed loop system (5), a common Lyapunov function is considered

$$
V = \hat{x}^T P \hat{x},
$$

(6)

where $P$ is a $(n + k) \times (n + k)$ fixed symmetric positive-definite matrix. $n$ is the number of states of the plant, and $k$ is the order of the controller.

Recently it is shown that the output feedback control problem can be changed into an LMI problem\[28]. Here we will follow the idea to solve our $H_{\infty}$ output feedback fuzzy control problem. For using LMI techniques, partition the common Lyapunov function $P$ and $P^{-1}$ as

$$
P = \begin{bmatrix} X & Y \\
N^T & N
\end{bmatrix}, \quad P^{-1} = \begin{bmatrix} M & X \\
N & N^T
\end{bmatrix},
$$

(7)

where $X$ and $Y$ have dimensions $n \times n$ and are symmetric. From $PP^{-1} = I$, we infer $P \begin{bmatrix} X \\
M^T
\end{bmatrix} = \begin{bmatrix} I \\
0
\end{bmatrix}$ which leads to

$$
P \Pi_1 = \Pi_2 \text{ with } \Pi_1 = \begin{bmatrix} X & I \\
M^T & 0
\end{bmatrix}, \quad \Pi_2 = \begin{bmatrix} I & Y \\
0 & N^T
\end{bmatrix}.
$$

(8)

Then we have the main result in this section.

**Theorem 1.** Given a fuzzy dynamic system described as in (2), an output feedback fuzzy controller described as in (4) and a constant $\gamma > 0$, if there exists a set of variables $\tilde{A}_{li}, \tilde{B}_i, \tilde{C}_i, X$ and $Y$ satisfying the following inequalities,

$$
\begin{bmatrix}
\Omega_{li11} & \Omega_{li21} & B_{wi} & (C_{zi}X)^T \\
\Omega_{li21} & \Omega_{li22} & YB_{wl} & C_{zi}^T \\
B_{wi}^T & B_{wl}^T Y & -\gamma^2 I & 0 \\
C_{zi}X & C_{zi} & 0 & -I
\end{bmatrix} < 0,
$$

(9)
\[
\begin{bmatrix}
X & I \\
I & Y
\end{bmatrix} > 0,
\]

where
\[
\Omega_{ii1} = A_iX + XA_i^T + B_i\bar{C}_i + (B_i\bar{C}_i)^T,\quad \Omega_{ii2} = \bar{A}_i + A_i^T,
\]
\[
\Omega_{ii2} = A_i^TY + YA_i + \bar{B}_i\bar{C}_i + (\bar{B}_i\bar{C}_i)^T;\quad l, i = 1, 2, \ldots, m,
\]

then the closed loop system (5) is asymptotically stable with \(H_\infty\) performance bound \(\gamma\).

Moreover, determining \(M, N\) by
\[
MN^T = I - XY,
\]
the parameters of the \(H_\infty\) output feedback controller are given by
\[
C_{cl} = \bar{C}_iM^T, \quad B_{cl} = N^{-1}\bar{B}_i, \quad A_{cl} = N^{-1}[\bar{A}_i - NB_{cl}C_iX - YB_{cl}C_iM^T - YA_iX]M^T.
\]

Proof. Firstly, it will be shown that (9) implies the following inequality
\[
\bar{A}^T(\mu)P + P\bar{A}(\mu) + \gamma^{-2}P\bar{B}(\mu)\bar{B}^T(\mu)P + \bar{C}^T(\mu)\bar{C}(\mu) < 0.
\]

Since
\[
\bar{A}^T(\mu)P + P\bar{A}(\mu) + \gamma^{-2}P\bar{B}(\mu)\bar{B}^T(\mu)P + \bar{C}^T(\mu)\bar{C}(\mu) =
\]
\[
\begin{bmatrix}
\sum_{l=1}^m \mu_l A_l & \sum_{l=1}^m \mu_l B_l & \sum_{l=1}^m \mu_l C_{ci} \\
\sum_{l=1}^m \mu_l B_{cl} & \sum_{l=1}^m \mu_l C_l & \sum_{l=1}^m \mu_l A_{ci} \\
\sum_{l=1}^m \mu_l B_{wl} & \sum_{l=1}^m \mu_l C_{zl} & 0
\end{bmatrix}^T P + P
\]
\[
\begin{bmatrix}
\sum_{l=1}^m \mu_l A_l & \sum_{l=1}^m \mu_l B_l & \sum_{l=1}^m \mu_l C_{ci} \\
\sum_{l=1}^m \mu_l B_{cl} & \sum_{l=1}^m \mu_l C_l & \sum_{l=1}^m \mu_l A_{ci} \\
\sum_{l=1}^m \mu_l B_{wl} & \sum_{l=1}^m \mu_l C_{zl} & 0
\end{bmatrix}
\]
\[
\gamma^{-2} P \begin{bmatrix}
\sum_{l=1}^m \mu_l B_{wl} \\
\sum_{l=1}^m \mu_l C_{zl} \\
0
\end{bmatrix}^T + \begin{bmatrix}
\sum_{l=1}^m \mu_l C_{zl}
\end{bmatrix}^T,
\]

it is easy to see that if the following set of inequalities is satisfied, then (13) is established,
\[
\begin{bmatrix}
A_i & B_iC_{ci} \\
B_{cl}C_l & A_{ci}
\end{bmatrix}^T P + P \begin{bmatrix}
A_i & B_iC_{ci} \\
B_{cl}C_l & A_{ci}
\end{bmatrix} + \gamma^{-2} P \begin{bmatrix}
B_{wl} \\
0
\end{bmatrix} \begin{bmatrix}
B_{wl} \\
0
\end{bmatrix}^T P + [C_{zl} 0]^T [C_{zl} 0] < 0,
\]
\[
l, i = 1, 2, \ldots, m,
\]

where the membership function property (3b) has been used.

Using Schur complement, (15) is equivalent to
\[
\begin{bmatrix}
\bar{A}_i^T P + P\bar{A}_i & \bar{P}\bar{B}_i & \bar{C}_i^T P \\
\bar{B}_i^T P & -\gamma^2 I & 0 \\
\bar{C}_i & 0 & -I
\end{bmatrix} < 0, \quad l, i = 1, 2, \ldots, m,
\]

where
\[
\bar{A}_i = \begin{bmatrix}
A_i & B_iC_{ci} \\
B_{cl}C_l & A_{ci}
\end{bmatrix}, \quad \bar{B}_i = \begin{bmatrix}
B_{wl} \\
0
\end{bmatrix}, \quad \bar{C}_i = [C_{zl} 0].
\]

Multiplying \(\text{diag}[I I I]\) on both sides of (16), we have
\[
\begin{bmatrix}
\Omega_{ii1} & \Omega_{ii2} & B_{wl} & (C_{zl}X)^T \\
\Omega_{ii21} & \Omega_{ii22} & YB_{wl} & \bar{C}_l^T \\
\bar{B}_i^T PII_l & -\gamma^2 I & 0 & -I \\
\bar{C}_l & 0 & -I
\end{bmatrix} < 0,
\]

where
\[
\Omega_{ii1} = A_iX + XA_i^T + B_i\bar{C}_i + (B_i\bar{C}_i)^T, \quad \Omega_{ii2} = \bar{A}_i + A_i^T,
\]
\[
\Omega_{ii2} = A_i^TY + YA_i + \bar{B}_i\bar{C}_i + (\bar{B}_i\bar{C}_i)^T; \quad l, i = 1, 2, \ldots, m.
\]

where

\[ \begin{align*}
\dot{\tilde{X}}_{i11} &= A_t X + X A_t^T + B_t C_{ci} M^T + [B_t C_{ci} M^T]_t, \\
\dot{\tilde{X}}_{i21} &= -N A_{ci} M^T + N B_{ci} C_i X + Y B_t C_{ci} M^T + Y A_t X + A_t^T, \\
\dot{\tilde{X}}_{i22} &= A_t^T Y + Y A_t + N B_{ci} C_i + [N B_{ci} C_i]_t.
\end{align*} \]

Now we can define the change formula of the output feedback controller variables for fuzzy dynamic systems,

\[ \tilde{A}_t = N A_{ci} M^T + N B_{ci} C_i X + Y B_t C_{ci} M^T + Y A_t X, \tilde{B}_t = -N D_{ci}, \tilde{C}_i = C_{ci} M^T. \]  (18)

Then substituting the variable change formula (18) into (17), we obtain (9).

Furthermore, multiply \( \Pi_1 \) on both sides of \( P > 0 \), that is,

\[ \Pi_1^T P \Pi_1 - \begin{bmatrix} X & I \\ I & Y \end{bmatrix} > 0, \]

the condition (10) is equivalent to \( P > 0 \). Then, it follows from (13) that the closed loop fuzzy control system is asymptotically stable, since (13) implies \( \tilde{A}_t^T (\mu) P + P \tilde{A}_t (\mu) < 0 \).

Now let us prove that the closed loop fuzzy control system satisfies the \( H_\infty \) performance. Differentiating the Lyapunov function (6),

\[ \dot{V}(t) = \left( \frac{d}{dt} \tilde{x}^T \right) P \tilde{x} + \tilde{x}^T P \left( \frac{d}{dt} \tilde{x} \right) = \tilde{x}^T \tilde{A}_t^T P \tilde{x} + \tilde{x}^T P \tilde{A}_t \tilde{x} + \tilde{x}^T P \tilde{B}_t \tilde{w} + \tilde{w}^T \tilde{B}_t \tilde{P} \tilde{x}. \]

Integrating the above function,

\[ V(\infty) - V(0) = \int_0^\infty [\tilde{x}^T (\tilde{A}_t^T P + P \tilde{A}_t) \tilde{x} + \tilde{x}^T P \tilde{B}_t \tilde{P} \tilde{x} + \tilde{x}^T P \tilde{B}_t \tilde{w} \tilde{x}] dt < \]

\[ \int_0^\infty [\tilde{x}^T (-\gamma^{-2} P \tilde{B}_t \tilde{B}_t^T P - \tilde{C}^T \tilde{C}) \tilde{x} + \tilde{x}^T P \tilde{B}_t \tilde{P} \tilde{x} + \tilde{x}^T P \tilde{B}_t \tilde{w} \tilde{x}] dt = \]

\[ \int_0^\infty [-z^T z + \gamma^2 w^T w - (w^T - \gamma^{-2} \tilde{x}^T P \tilde{B}_t \tilde{w}) \gamma^2 (w - \gamma^{-2} \tilde{B}_t^T P \tilde{x})] dt \leq \]

\[ \int_0^\infty [-z^T z + \gamma^2 w^T w] dt \]

where (13) has been used.

Therefore, with zero initial condition, it follows that the stable global closed loop system (5) satisfies the \( H_\infty \) performance, that is, \( \|z\| \leq \gamma \|w\| \). \( \square \)

In light of the above theorem, the following procedure to evaluate an \( H_\infty \) output feedback control law is developed.

**Algorithm 1 of the \( H_\infty \) fuzzy output feedback control design:**

Step 1. Choose an \( H_\infty \) performance bound \( \gamma > 0 \).

Step 2. Use the LMI toolbox in Matlab to solve the inequalities (9) and (10). If the solutions to \( \tilde{A}_t, \tilde{B}_t, \tilde{C}_i, X \) and \( Y, l, i = 1, 2, \ldots, m \) are found, then determine \( M, N \) to satisfy (11), and an output feedback controller can be obtained by (12), and stop. Otherwise declare that the proposed algorithm fails.

It is noted that the above \( H_\infty \) output feedback fuzzy control design is based on the assumption that there exists a common Lyapunov function for fuzzy dynamic systems. In many cases, it is very hard to find such a common Lyapunov function for fuzzy dynamic systems or
4 \( H_\infty \) Output Feedback Control Design based on a Piecewise Differential Lyapunov Function

We will develop a new \( H_\infty \) output feedback fuzzy control design method based on a piecewise differential Lyapunov function. For the control design purpose, we decompose the state space into \( m \) independent regions which are defined as follows

\[
D_l = \{ q | \mu_i(q) \geq \mu_j(q), \quad j = 1, 2, \cdots, m, j \neq l \} \quad \forall \mu \in M, l = 1, 2, \cdots, m. \quad (19)
\]

The boundary of \( D_l \) is denoted by

\[
\partial D_l = \{ q | \mu_i(q) = \mu_j(q), \quad D_l \cap D_j \neq 0, \quad j = 1, 2, \cdots, m, j \neq l \}, \forall \mu \in M, l = 1, 2, \cdots, m. \quad (20)
\]

Then, on every region \( D_l \) the fuzzy system (2) can be expressed by

\[
\begin{align*}
\dot{x}(t) &= (A_l + \Delta A_l(\mu))x(t) + (B_l + \Delta B_l(\mu))u(t) + (B_{wl} + \Delta B_{wl}(\mu))w(t), \\
y(t) &= (C_l + \Delta C_l(\mu))x(t), \\
z(t) &= (C_{zl} + \Delta C_{zl}(\mu))x(t),
\end{align*}
\]

where

\[
\begin{align*}
\Delta A_l(\mu) &= \sum_{i=1, i \neq l}^m \mu_i \Delta A_{li}, \\
\Delta B_l(\mu) &= \sum_{i=1, i \neq l}^m \mu_i \Delta B_{li}, \\
\Delta B_{wl}(\mu) &= \sum_{i=1, i \neq l}^m \mu_i \Delta B_{wli}, \\
\Delta C_l(\mu) &= \sum_{i=1, i \neq l}^m \mu_i \Delta C_{li}, \\
\Delta C_{zl}(\mu) &= \sum_{i=1, i \neq l}^m \mu_i \Delta C_{zli}, \\
\Delta A_{li} &= A_{li} - A_l, \\
\Delta B_{li} &= B_{li} - B_l, \\
\Delta B_{wli} &= B_{wli} - B_{wl}, \\
\Delta C_{li} &= C_{li} - C_l, \\
\Delta C_{zli} &= C_{zli} - C_{zl}, \\
\forall \mu \in M, \quad q(t) \in D_l, \quad l = 1, 2, \cdots, m.
\end{align*}
\]

Here the \( l \)-th local system is different from the fuzzy dynamic local model in (2) because it considers all interactions among the local models. We assume that if the system is in the \( l \)-th state region, it will stay there for a period of time \( \bar{t} \) where \( \bar{t} = \tau_{l+1} - \tau_l, i = 1, 2, \cdots, \bar{T}, \tau_i \) is the \( i \)-th time instant at which the \( q(t) \) is on a boundary \( \partial D_l \). \( \bar{T} \) is the total number of transits among regions.

Consider a piecewise continuous output feedback controller below,

\[
K_l : \begin{cases}
\dot{x}_c - A_{cl}x_c(t) + B_{cl}y(t), \\
u = C_{cl}x_c(t).
\end{cases} \quad q(t) \in D_l, \quad t \neq \tau_l, \quad l = 1, 2, \cdots, m. \quad (22)
\]

With the output feedback controller (22), on every region \( D_l \) the closed loop fuzzy control system can be described as

\[
\begin{align*}
\dot{\bar{x}}(t) &= \dot{\bar{A}}_l(\mu)\bar{x}(t) + \dot{\bar{B}}_l(\mu)w(t) = (\dot{\bar{A}}_l + \Delta \dot{\bar{A}}_l(\mu))\bar{x}(t) + (\dot{\bar{B}}_l + \Delta \dot{\bar{B}}_l(\mu))w(t), \\
z(t) &= \dot{\bar{C}}_l(\mu)\bar{x}(t) = (\dot{\bar{C}}_{zl} + \Delta \dot{\bar{C}}_{zl}(\mu))\bar{x}(t),
\end{align*}
\]

where

\[
\begin{align*}
\dot{\bar{A}}_l &= \begin{bmatrix} A_l & B_{cl}C_{zl} \\ B_{cl}C_l & A_{cl} \end{bmatrix}, \\
\Delta \dot{\bar{A}}_l(\mu) &= \begin{bmatrix} \Delta A_l & \Delta B_{cl}C_{zl} \\ \Delta B_{cl}C_l & 0 \end{bmatrix}, \\
\dot{\bar{B}}_l &= \begin{bmatrix} B_{wl} \\ 0 \end{bmatrix}, \\
\Delta \dot{\bar{B}}_l(\mu) &= \begin{bmatrix} \Delta B_{wl} \\ 0 \end{bmatrix}, \\
\dot{\bar{C}}_{zl} &= \begin{bmatrix} C_{zl} & 0 \end{bmatrix}, \\
\Delta \dot{\bar{C}}_{zl}(\mu) &= \begin{bmatrix} \Delta C_{zl} & 0 \end{bmatrix}, \\
\bar{x} &= \begin{bmatrix} x \\ x_c \end{bmatrix}, \quad q(t) \in D_l, \quad \forall \mu \in M, \quad l = 1, 2, \cdots, m.
\end{align*}
\]
We assume that the state $\tilde{x}(t) \in R^{n+k}$ does not jump at the time instant $\tau_i$, that is,

$$\tilde{x}(\tau^-_i) = \tilde{x}(\tau_i) = \tilde{x}(\tau^+_i), \quad i = 1, 2, \cdots, T. \tag{24}$$

Consider the following piecewise differentiable Lyapunov function

$$V(t) = \tilde{x}^TP(t)\tilde{x}, \quad P(t) = P_1, \quad q(t) \in D_1, t \neq \tau_i, \tag{25}$$

$$P(\tau^-_i) = P_1, \quad q(\tau^-_i) \in \partial D_1, \quad P(\tau^+_i) = P_2, \quad q(\tau^+_i) \in D_2, \quad D_1 \cap D_2 \neq 0, t = \tau_i, \tag{26}$$

$$j, l - 1, 2, \cdots, m, j \neq i, i - 1, 2, \cdots, T,$$

where $(P_1, P_2, \cdots, P_m)$ is a set of $(n \times k) \times (n \times k)$ fixed symmetric positive-definite matrices and $P(t)$ is a symmetric positive-definite piecewise differentiable matrix function. The following notations are used for $P(t)$.

The left limit of $P(t)$ at $t \in R$ is defined as $P(t^-) = \lim_{\varepsilon \rightarrow 0^+} P(t - \varepsilon)$ if the limit exists; and the right limit of $P(t)$ at $t \in R$ is defined as $P(t^+) = \lim_{\varepsilon \rightarrow 0} P(t + \varepsilon)$ if the limit exists. Then we refer to a lemma from [10] about stabilization of fuzzy dynamic control systems.

**Lemma 1.** The closed loop fuzzy system is globally asymptotically stable with $H_\infty$ performance bound $\gamma$, if there exist a controller and a positive-definite symmetric matrix function $P(t)$ such that

$$\dot{A}_l^T(\mu)P(t) + P(t)\dot{A}_l(\mu) + \gamma^2P(t)\tilde{B}_l(\mu)\tilde{B}_l^T(\mu)P(t) + \tilde{C}_l^T(\mu)\tilde{C}_l(\mu) < 0, \tag{27}$$

$$P(\tau^-_i) \leq P(\tau^-_i), \quad i = 1, 2, \cdots, T, \quad \forall \mu \in M \text{ and } t \in [0, \infty). \tag{28}$$

For the control design purpose, the following upper bounds for the uncertainties of the fuzzy local systems (21) are defined, which can be found as in [7,9]

$$\Delta A_l^T(\mu)\Delta A_l(\mu) \leq E_l^T E_l, \quad \Delta B_{wl} \Delta B_{wl}^T \leq E_{B_{wl}} \dot{E}_{B_{wl}}, \quad \Delta C_{zl}^T(\mu)\Delta C_{zl}(\mu) \leq E_{C_{zl}}^T E_{C_{zl}}, \tag{29}$$

where

$$\Delta A_l(\mu) = [\Delta A_l, \Delta B_{zl}], \quad E_l = [E_{A_l} \quad E_{B_{wl}}], \quad l = 1, 2, \cdots, m.$$

For using LMI techniques, partition the piecewise differentiable Lyapunov function $P_l$ and $P_{l}^{-1}$ as

$$P_l = \begin{bmatrix} Y_l & N_l \\ N_l^T & \ast \end{bmatrix}, \quad P_{l}^{-1} = \begin{bmatrix} X_l & M_l \\ M_l^T & \ast \end{bmatrix}, \quad l = 1, 2, \cdots, m, \tag{30}$$

where $X_l$ and $Y_l$ have dimensions $n \times n$ and are symmetric. From $P_l P_{l}^{-1} = I$, we infer

$$P_l \begin{bmatrix} X_l & M_l^T \\ \ast & \ast \end{bmatrix} = \begin{bmatrix} I \\ 0 \end{bmatrix} \quad \text{which leads to} \quad P_l \Pi_{l1} = \Pi_{l2} \quad \text{with} \quad \Pi_{l1} = \begin{bmatrix} X_l & I \\ M_l^T & 0 \end{bmatrix}, \quad \Pi_{l2} = \begin{bmatrix} I & Y_l \\ 0 & N_l^T \end{bmatrix}, \quad l = 1, 2, \cdots, m. \tag{31}$$

Here we refer to a lemma in [24] for the later use.

**Lemma 2.** For any constant $\varepsilon > 0$ and any matrices $X$ and $Y$ with appropriate dimensions, we have $X^T Y + Y^T X \leq \varepsilon X^T X + \varepsilon^{-1} Y^T Y$.

Then we have the following result in this section.

**Theorem 2.** Given a fuzzy dynamic system described as in (21), an output feedback controller described as in (22), the approximate upper bounds as in (29) and a constant $\gamma > 0$, if there exist a set of constants $\varepsilon_l > 0$, and a set of variables $A_l, B_l, C_l, E_{l0}, E_{l1}, E_{l2}, X_l$ and $Y_l$
satisfying the following inequalities,

\[
\begin{bmatrix}
\Omega_{111} & \Omega_{121}^T & X_t & I & \Omega_{131}^T & \Omega_{141}^T & E_{10}^T & B_{ul} & E_{Bul} & (C_{sl} X_t)^T & (E_{Csl} X_t)^T \\
\Omega_{121} & \Omega_{122} & I & Y_t & E_{Al} & (Y_t E_{Al})^T & Y_t B_{ul} & Y_t E_{Bul} & C_{sl}^T & E_{Csl}^T \\
X_t & I & -\epsilon_t^{-1} E_{11} & -\epsilon_t^{-1} X_t & 0 & 0 & 0 & 0 & 0 & 0 \\
X_t & I & -\epsilon_t^{-1} X_t & -\epsilon_t^{-1} Y_t & 0 & 0 & 0 & 0 & 0 & 0 \\
Y_t & I & -\epsilon_t^{-1} Y_t & -\epsilon_t^{-1} Y_t & 0 & 0 & 0 & 0 & 0 & 0 \\
Y_t & I & -\epsilon_t^{-1} Y_t & -\epsilon_t^{-1} Y_t & 0 & 0 & 0 & 0 & 0 & 0 \\
E_{Al} & Y_t E_{Al} & 0 & 0 & -\epsilon_t Y_t & -\epsilon_t Y_t & 0 & 0 & 0 & 0 \\
B_{ul}^T & B_{ul}^T Y_t & 0 & 0 & 0 & 0 & -2^{-1} \gamma^2 I & 0 & 0 & 0 \\
E_{Bul}^T & E_{Bul}^T Y_t & 0 & 0 & 0 & 0 & 0 & -2^{-1} \gamma^2 I & 0 & 0 \\
C_{sl} X_t & C_{sl} & 0 & 0 & 0 & 0 & 0 & 0 & 2^{-1} I & 0 \\
E_{Csl} X_t & E_{Csl} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2^{-1} I \\
\end{bmatrix} < 0,
\]

(32)

and the boundary condition

\[P(\tau_i^+) \leq P(\tau_i^-), \quad i = 1, 2, \ldots, \bar{T},\]

(34)

where

\[
\Omega_{111} = A_t X_t \quad \Omega_{121} = X_t A_t^T + B_t \bar{C}_t \quad (B_t \bar{C}_t)^T, \quad \Omega_{131} = A_t^T, \\
\Omega_{122} = A_t^T Y_t + Y_t A_t + \bar{B}_t C_t + (\bar{B}_t C_t)^T, \quad \Omega_{132} = E_{Al} X_t + E_{Bul} C_t, \quad l = 1, 2, \ldots, m,
\]

then the closed loop system (23) is asymptotically stable with \(H_\infty\) performance bound \(\gamma\). And determining \(M_t, N_t^T\) by

\[M_t N_t^T = I - X_t Y_t,\]

(35)

the parameters of the \(H_\infty\) output feedback controller are given by

\[C_{cl} = C_t M_t^{-T}, B_{cl} = N_t^{-1} B_t, A_{cl} = N_t^{-1} [A_t - N_t R_t C_t X_t - Y_t R_t C_{cl} M_t^{-T} - Y_t A_t X_t] M_t^{-T}.\]  

(36)

Proof. Firstly, it will be shown that (32) implies the inequality (27) in Lemma 1.

\[
\bar{A}_t^T(\rho) P(t) + P(t) \tilde{A}_t(\rho) + \gamma^{-2} P(t) \tilde{B}_t(\rho) \tilde{B}_t^T(\rho) P(t) + \tilde{C}_t^T(\rho) \tilde{C}_t(\rho) P(t) \\
\leq A_t^T P_t + P_t A_t + \epsilon_t P_t + \epsilon_t^{-1} \Delta A_t^T \Delta A_t + 2\gamma^{-2} P_t \tilde{B}_ul \tilde{B}_ul^T P_t + 2\gamma^{-2} P_t \Delta \tilde{B}_ul \Delta \tilde{B}_ul^T P_t + 2\Delta \tilde{C}_t^T \Delta \tilde{C}_t \\
= A_t^T P_t + P_t A_t + \epsilon_t P_t + \epsilon_t^{-1} \Delta A_t^T \Delta A_t + 2\gamma^{-2} P_t \tilde{B}_ul \tilde{B}_ul^T P_t + 2\Delta \tilde{C}_t^T \tilde{C}_t + 2[\Delta C_{cl}^T 0]^T[\Delta C_{cl} 0]
\]

(37)

where the lemma 2 has been used.
Thus if the right hand side of (37) is less than zero, we can show (27). Using Schur complement, the right hand side of (37) being less than zero is equivalent to

\[
\begin{bmatrix}
\tilde{A}_T^T P_l + P_l \tilde{A}_t & P_l \Sigma_{Al} & P_l \tilde{B}_{wl} & P_l \Sigma_{Bwl} & \tilde{C}_l^T & \Sigma_{Cl}^T \\
\Sigma_{Fl} & c^{-1}I & 0 & 0 & 0 & 0 \\
\Sigma_{Al} & 0 & -\varepsilon I & 0 & 0 & 0 \\
\tilde{B}_{wl}^T P_l & 0 & 0 & 0 & 0 & 0 \\
\Sigma_{Bwl} & 0 & 0 & 0 & 0 & 0 \\
\tilde{C}_l & 0 & 0 & 0 & 0 & 0 \\
\Sigma_{Cl} & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix} < 0,
\]

(38)

where

\[
\Sigma_{Al} = \begin{bmatrix} E_{Al} & \varepsilon B_{I} \tilde{C}_{cl} \\ 0 & 0 \end{bmatrix}, \quad \Sigma_{Bwl} = \begin{bmatrix} E_{Bwl} \\ 0 \end{bmatrix}, \quad \Sigma_{Cl} = \begin{bmatrix} E_{C_{cl}} & 0 \end{bmatrix}.
\]

Multiplying diag[\(\Pi_{11} \Pi_{11} \Pi_{12} \ I \ I \ I\)] on both sides of (38), we have

\[
\begin{bmatrix}
\tilde{\Omega}_{111} & \tilde{\Omega}_{112}^T & X_t & I & \tilde{\Omega}_{113} & \tilde{\Omega}_{114}^T & B_{wl} & E_{Bwl} & (C_{T_{cl}} X_t)^T & (E_{C_{cl}} X_t)^T \\
\tilde{\Omega}_{121} & \tilde{\Omega}_{122} & I & Y_t & E_{A_{cl}} & \tilde{\Omega}_{123} & Y_t B_{wl} & Y_t E_{Bwl} & C_{T_{cl}} & E_{C_{cl}} \\
X_t & I & \tilde{\Omega}_{133} & -\varepsilon^{-1} I X_t & 0 & 0 & 0 & 0 & 0 & 0 \\
I & Y_t & -\varepsilon^{-1} I X_t & -\varepsilon^{-1} I & 0 & 0 & 0 & 0 & 0 & 0 \\
\tilde{\Omega}_{151} & \tilde{\Omega}_{152} & E_{A_{cl}} & 0 & 0 & -\varepsilon I t & -\varepsilon y_t & 0 & 0 & 0 \\
\tilde{\Omega}_{161} & \tilde{\Omega}_{162} & 0 & 0 & -\varepsilon y_t & \tilde{\Omega}_{163} & 0 & 0 & 0 & 0 \\
B_{wl}^T & B_{wl}^T & Y_t & 0 & 0 & 0 & 0 & -\varepsilon^{-1} \gamma^2 I & 0 & 0 \\
E_{Bwl}^T & E_{Bwl}^T & Y_t & 0 & 0 & 0 & 0 & 0 & -\varepsilon^{-1} \gamma^2 I & 0 \\
C_{cl} X_t & C_{cl} & 0 & 0 & 0 & 0 & 0 & 0 & -\varepsilon I t Y_t & -\varepsilon I (Y_t Y_t + N_t X_t) \\
F_{C_{cl}} & F_{C_{cl}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\varepsilon I (Y_t Y_t + N_t X_t) \end{bmatrix} < 0,
\]

(39)

where

\[
\tilde{\Omega}_{111} = A_t X_t + X_t A_t^T + B_t C_{cl} M_t^T + \left[B_t C_{cl} M_t^T\right]^T, \tilde{\Omega}_{122} = A_t^T Y_t + Y_t A_t + N_t B_{cl} C_{cl} + \left[N_t B_{cl} C_{cl}\right]^T, \\
\tilde{\Omega}_{121} = A_t X_t + X_t A_t^T + B_t C_{cl} M_t^T + \left[B_t C_{cl} M_t^T\right]^T, \tilde{\Omega}_{123} = -\varepsilon^{-1} (X_t X_t + M_t M_t^T), \\
\tilde{\Omega}_{151} = E_{A_{cl}} X_t + E_{B_{cl}} (C_{cl} M_t^T), \tilde{\Omega}_{161} = Y_t E_{B_{cl}} C_{cl} M_t^T + Y_t E_{A_{cl}} X_t, \\
\tilde{\Omega}_{162} = Y_t E_{A_{cl}}, \tilde{\Omega}_{163} = -\varepsilon^{-1} Y_t Y_t + N_t X_t.
\]

Now we can define the change formula of the output feedback controller variables for fuzzy dynamic systems,

\[
\tilde{A}_t = N_t A_{cl} M_t^T + N_t B_{cl} C_{cl} X_t + Y_t B_t C_{cl} M_t^T + Y_t A_t X_t, \tilde{B}_t = N_t B_{cl} C_{cl} M_t^T, \\
\tilde{E}_{t0} = Y_t E_{B_{cl}} C_{cl} M_t^T + Y_t E_{A_{cl}} X_t, \tilde{E}_1 = X_t X_t + M_t M_t^T, \tilde{E}_2 = Y_t Y_t + N_t Y_t.
\]

(40)

Then substituting the variable change formula (40) into (39), we obtain (32). Furthermore, multiply \(\Pi_{11}\) on both sides of \(P_l > 0\), that is,

\[
\Pi_{11}^T P_l \Pi_{11} = \begin{bmatrix} X_t & I \\ I & Y_t \end{bmatrix} > 0,
\]

the condition (33) is equivalent to \(P_l > 0\).

Then it follows from Lemma 1 that the closed loop fuzzy control system is asymptotically stable.
Now let us show $H_\infty$ performance. Differentiating $\tilde{\alpha}^T P \tilde{\alpha}$ and then integrating form $\tau_i^+$ to $\tau_{i+1}$ we have

$$\int_{\tau_i^+}^{\tau_{i+1}} \left( \frac{d}{dt} (\tilde{\alpha}^T P \tilde{\alpha}) \right) dt =$$

$$\int_{\tau_i^+}^{\tau_{i+1}} [\tilde{\alpha}^T (\tilde{A}_i^T P + P \tilde{A}_i) \tilde{\alpha} + \tilde{w}^T \tilde{B}_i^T P \tilde{\alpha} + \tilde{\alpha}^T P \tilde{B}_i \tilde{w} \tilde{\alpha}] dt <$$

$$\int_{\tau_i^+}^{\tau_{i+1}} [\tilde{\alpha}^T (-\gamma^{-2} P \tilde{B}_i \tilde{B}_i^T P - \tilde{C}_i \tilde{C}_i^T P \tilde{\alpha} + \tilde{w}^T \tilde{B}_i^T P \tilde{\alpha} + \tilde{\alpha}^T P \tilde{B}_i \tilde{w} \tilde{\alpha}] dt =$$

$$\int_{\tau_i^+}^{\tau_{i+1}} [-z^T z + \gamma^2 w^T w - (w^T - \gamma^{-2} \tilde{\alpha}^T P \tilde{B}_i(t)) \gamma^2 (w - \gamma^{-2} \tilde{B}_i(t) P \tilde{\alpha})] dt,$$

where the proved condition (27) has been used. Thus we obtain

$$\tilde{\alpha}^T P(\infty) \tilde{\alpha} - \tilde{\alpha}^T P(0) \tilde{\alpha} <$$

$$\int_0^\infty \left[ -z^T z + \gamma^2 w^T w - (w^T - \gamma^{-2} \tilde{\alpha}^T P \tilde{B}(t)) \gamma^2 (w - \gamma^{-2} \tilde{B}(t) P \tilde{\alpha}) \right] dt +$$

$$\sum_{i} [\tilde{\alpha}^T P(\tau_i^+) \tilde{\alpha} - \tilde{\alpha}^T P(\tau_i^-) \tilde{\alpha}] <$$

$$\int_0^\infty [-z^T z + \gamma^2 w^T w] dt + \sum_{i} [\tilde{\alpha}^T P(\tau_i^+) \tilde{\alpha} - \tilde{\alpha}^T P(\tau_i^-) \tilde{\alpha}]. \quad (41)$$

From (34) we have $\sum_{i=0}^{T} [\tilde{\alpha}^T P(\tau_i^+) \tilde{\alpha} - \tilde{\alpha}^T P(\tau_i^-) \tilde{\alpha}] \leq 0$.

Therefore, with zero initial condition, it follows that the stable global closed loop system (23) satisfies the $H_\infty$ performance, that is, $\|z\| \leq \gamma \|w\|$. □

Based on the above theorem, an algorithm to obtain an $H_\infty$ output feedback controller is as follows.

**Algorithm 2 of the $H_\infty$ fuzzy output feedback control design:**

Step 1. Choose the approximate upper bounds (29) by using nonlinear programming algorithm or experience in [7] and a constant $\gamma$.

Step 2. Set $\varepsilon_l = 1$, for $l = 1, 2, \cdots, m$.

Step 3. Use the LMI toolbox in Matlab to solve the inequalities (34) and (35). If the solutions to $\tilde{A}_i, \tilde{B}_i, \tilde{C}_i, \tilde{E}_{10}, \tilde{E}_{11}, \tilde{E}_{12}, X_i$ and $Y_i$ are found, then determine $M_l, N_l$ to satisfy (37), and an $H_\infty$ output feedback controller can be obtained by (38), and stop. Otherwise go to the next step.

Step 4. Set $\varepsilon_l = \varepsilon_l/2$. If $\varepsilon_l$ is less than some prespecified computational threshold, then stop and declare that the proposed algorithm fails. Otherwise, go to Step 3.

5 An Example

To illustrate the $H_\infty$ fuzzy output feedback control design algorithms, we consider the following problem of balancing an inverted pendulum on a cart. The equations of motion for the pendulum are

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = \frac{g \sin(x_1) - amlx_2^2 \sin(2x_1)/2 - \cos(x_1)u}{4l/3 - aml \cos^2(x_1)} + w, \quad (42)$$
where \( x_1 \) denotes the angle of the pendulum from the vertical, \( x_2 \) is the angular velocity, \( g = 9.8 \text{ m/s}^2 \) is the gravity constant, \( w \) is the external disturbance variable which is a sinusoidal signal, \( w = \sin(2\pi t) \). \( m \) is the mass of the pendulum, \( M \) is the mass of the cart, \( 2l \) is the length of the pendulum, and \( u \) is the force applied to the cart. \( a = 1/(m + M) \). We choose \( m = 2.0 \text{ kg}, M = 8.0 \text{ kg}, 2l = 1.0 \text{ m} \) in the simulation. The following fuzzy model is used to design \( H_\infty \) output feedback fuzzy controllers.

\[
R^1: \text{IF } x_1(t) \text{ is about 0, } x_2(t) \text{ is about 0,} \\
\text{THEN } \dot{x}(t) = A_1 x(t) + B_1 u(t) + B_{w1} w(t), \\
y(t) = C_1 x(t), \\
z(t) = C_{z1} x(t),
\]

\[
R^2: \text{IF } x_1(t) \text{ is about } \pm \pi/3, x_2(t) \text{ is about 0,} \\
\text{THEN } \dot{x}(t) = A_2 x(t) + B_2 u(t) + B_{w2} w(t), \\
y(t) = C_2 x(t), \\
z(t) = C_{z2} x(t),
\]

where

\[
A_1 = \begin{bmatrix} 0 & 1 \\ 17.2941 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ -0.1765 \end{bmatrix}, \quad B_{w1} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C_1 = [1 \ 0], \quad C_{z1} = [1 \ 0];
\]

\[
A_2 = \begin{bmatrix} 0 & 1 \\ 12.6305 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ -0.0779 \end{bmatrix}, \quad B_{w2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C_2 = [1 \ 0], \quad C_{z2} = [1 \ 0].
\]

We use the following membership functions

\[
\mu_1(x_1(t)) = \frac{1}{1 + \exp(-7(x_1(t) - \pi/4)))} \cdot \frac{1}{1 + \exp(-7(x_1(t) + \pi/4)))},
\]

\[
\mu_2(x_1(t)) = 1 - \mu_1(x_1(t)).
\]

Application of Algorithm 1.

**Case 1** \((\gamma = \infty)\). Let the \( H_\infty \) performance bound \( \gamma = \infty \). According to the first algorithm, the following result has been obtained

\[
\tilde{A}_{11} = \begin{bmatrix} -90.1320 & -7910 \\ -7910 & 0.5188 \end{bmatrix}, \quad \tilde{B}_1 = \begin{bmatrix} -302.2 \\ 31 \end{bmatrix}, \quad \tilde{C}_1 = [302.8 \ 726.5],
\]

\[
\tilde{A}_{22} = \begin{bmatrix} -11.9122 & -4066 \\ -4066 & 0.4810 \end{bmatrix}, \quad \tilde{B}_2 = \begin{bmatrix} -302.2 \\ 31 \end{bmatrix}, \quad \tilde{C}_2 = [302.8 \ 726.5],
\]

\[
X = \begin{bmatrix} 1.2716 & -3.7912 \\ -3.7912 & 28.5691 \end{bmatrix}, \quad Y = \begin{bmatrix} 100.9821 & -2.6440 \\ -2.6440 & 0.9071 \end{bmatrix}.
\]

Choosing \( M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \), and computing \( N = \begin{bmatrix} -137.4 & 458.4 \\ 6.8 & -34.9 \end{bmatrix} \) by (11). For the first local system and the second local system respectively, the local output feedback controllers based on (12) have been obtained as follows

\[
A_{c1} = \begin{bmatrix} -521.0 & 679.3 \\ -92.15 & 402.1 \end{bmatrix}, \quad B_{c1} = \begin{bmatrix} 498.2041 \\ 217.2123 \end{bmatrix}, \quad C_{c1} = [231.1 \ 486.3];
\]

\[
A_{c2} = \begin{bmatrix} -582.9 & 786.2 \\ -84.60 & 390.4 \end{bmatrix}, \quad B_{c2} = \begin{bmatrix} 498.2041 \\ 217.2123 \end{bmatrix}, \quad C_{c2} = [231.1 \ 486.3].
\]

Fig.1 shows the angle response of the system (42) for an initial condition \( x_1 = 70^\circ, x_2 = 0.1t \) can be observed that the performance of disturbance attenuation is not satisfactory.
Case 2 ($\gamma = 1$). Now we choose the bound $\gamma = 1$. Similar to the procedure of Case 1, the following result has been also obtained

$$
\tilde{A}_{11} = \begin{bmatrix} -110.0881 & -0.6168 \\ -6168 & -9.160 \end{bmatrix}, \quad \tilde{B}_1 = \begin{bmatrix} -30256 \\ 26 \end{bmatrix}, \quad \tilde{C}_1 = [275.9 \quad 1392.4],
$$

$$
\tilde{A}_{22} = \begin{bmatrix} -83.0442 & -4.807 \\ -4.807 & -3.013 \end{bmatrix}, \quad \tilde{B}_2 = \begin{bmatrix} -30256 \\ 26 \end{bmatrix}, \quad \tilde{C}_2 = [275.9 \quad 1392.4],
$$

$$
X = \begin{bmatrix} 1.1705 & -4.2171 \\ -4.2171 & 39.4660 \end{bmatrix}, \quad Y = \begin{bmatrix} 215.5097 & -2.3192 \\ -2.3192 & 0.5490 \end{bmatrix}.
$$

With $M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $N = \begin{bmatrix} -261.0 & 1000.4 \\ -5.0 & -30.4 \end{bmatrix}$, the first and the second local controllers have been obtained respectively

$$
A_{c1} = \begin{bmatrix} -1604.9 & 3071.7 \\ -233.9 & 1014.0 \end{bmatrix}, \quad B_{c1} = \begin{bmatrix} 1020.9 \\ 324.1 \end{bmatrix}, \quad C_{c1} = [205.2 \quad 1188.3];
$$

$$
A_{c2} = \begin{bmatrix} -2249.5 & 2928.6 \\ -269.7 & 984.8 \end{bmatrix}, \quad B_{c2} = \begin{bmatrix} 1020.9 \\ 324.1 \end{bmatrix}, \quad C_{c2} = [205.2 \quad 1188.3].
$$

Fig.2 shows the angle response for the same initial condition. The performance of disturbance attenuation has been greatly improved.

**Application of Algorithm 2.**

Using the approximate upper bound searching algorithm in [7], we can find the upper bounds for the first local system and the second local system respectively,

$$
E_1 = [E_{A1} \quad E_{B1}]; \quad E_{A1} = 0.1(A_2 - A_1), \quad E_{B1} = 0.01(B_2 - B_1). \quad (45a)
$$

$$
F_2 = [F_{A2} \quad F_{B2}]; \quad E_{A2} = 0.1(A_1 - A_2), \quad E_{B2} = 0.01(B_1 - B_2). \quad (45b)
$$

Case 3 ($\gamma = \infty$). Let the $H_{\infty}$ performance bound $\gamma = \infty$. For the first local system, according to Algorithm 2, for $\varepsilon_1 = 0.1$, the following result has been obtained

$$
\tilde{A}_1 = \begin{bmatrix} -52.6031 & -0.5922 \\ -0.5922 & 0.3003 \end{bmatrix}, \quad \tilde{B}_1 = \begin{bmatrix} -1809 \\ 392.1 \end{bmatrix}, \quad \tilde{C}_1 = [270.8 \quad 659.1],
$$

$$
X_1 = \begin{bmatrix} 0.8211 & -2.3202 \\ -2.3202 & 35.1244 \end{bmatrix}, \quad Y_1 = \begin{bmatrix} 196.4483 & -3.6804 \\ -3.6804 & 0.5611 \end{bmatrix}.
$$

Choosing $M_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, and computing $N_1 = \begin{bmatrix} -168.8430 & 585.0712 \\ 4.3238 & -27.2476 \end{bmatrix}$ by (35), we can obtain the first local output feedback controller based on (36) as follows
\[ A_{o1} = \begin{bmatrix} -366.4508 & 866.2116 \\ -22.3366 & 93.2886 \end{bmatrix}, \quad B_{o1} = \begin{bmatrix} 378.9067 \\ 52.0554 \end{bmatrix}, \quad C_{o1} = [270.8 \quad 659.1]. \]

For the second local system, for \( \varepsilon_2 = 0.1 \), the following result has been obtained

\[ A_2 = \begin{bmatrix} -29.0099 & -.8601 \\ -.8601 & -.8092 \end{bmatrix}, \quad B_2 = \begin{bmatrix} -497.8 \\ 6.3 \end{bmatrix}, \quad C_2 = [235.8507 \quad 799.0445]. \]

\[ X_2 = \begin{bmatrix} .9132 & -2.1156 \\ -2.1156 & 18.5530 \end{bmatrix}, \quad Y_2 = \begin{bmatrix} 88.2243 & -3.3092 \\ -3.3092 & 1.0091 \end{bmatrix}. \]

Selecting \( M_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \) and finding \( N_2 = \begin{bmatrix} -86.5764 & 248.0429 \\ 5.1568 & -24.7228 \end{bmatrix} \) by (35), we can obtain the second local output feedback controller based on (36) as follows

\[ A_{o2} = \begin{bmatrix} -181.2207 & 328.0026 \\ -36.1109 & 53.0203 \end{bmatrix}, \quad B_{o2} = \begin{bmatrix} 280.2263 \\ 57.0036 \end{bmatrix}, \quad C_{o2} = [235.8507 \quad 799.0445]. \]

Fig. 3 shows the angle response of the system (42) for the initial condition \( x_1 = 70^\circ, x_2 = 0 \). It can be observed that the performance of disturbance attenuation is not satisfactory.

**Case 4 (\( \gamma = 1 \)).** Now we choose the bound \( \gamma = 1 \). Similar to Case 3, for the first local system, for \( \varepsilon_1 = 0.1 \), the following result has been obtained

\[ A_1 = \begin{bmatrix} -25.4997 & -.5922 \\ -5.922 & 1.824 \end{bmatrix}, \quad \tilde{B}_1 = \begin{bmatrix} -4908.3 \\ -11.2 \end{bmatrix}, \quad \tilde{C}_1 = [231.0 \quad 1970.5], \]

\[ X_1 = \begin{bmatrix} .6008 & -1.9942 \\ -1.9942 & 19.8816 \end{bmatrix}, \quad Y_1 = \begin{bmatrix} 112.0088 & -2.0147 \\ -2.0147 & .8447 \end{bmatrix}. \]

With \( M_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \) and \( N_1 = \begin{bmatrix} -70.3126 & 263.4234 \\ 2.8949 & -19.8117 \end{bmatrix} \) the first local controller as follows

\[ A_{c1} = \begin{bmatrix} -155.0308 & 354.3088 \\ -26.3354 & 28.0917 \end{bmatrix}, \quad B_{c1} = \begin{bmatrix} 278.7704 \\ 46.0198 \end{bmatrix}, \quad C_{c1} = [231.0 \quad 1970.5]. \]

For the second local system, for \( \varepsilon_2 = 0.1 \), result, has been found

\[ A_2 = \begin{bmatrix} -20.6622 & -.8038 \\ -.8038 & 1.8507 \end{bmatrix}, \quad \tilde{B}_2 = \begin{bmatrix} -3946.6 \\ 8.3 \end{bmatrix}, \quad \tilde{C}_2 = [258.9 \quad 920.2], \]

\[ X_2 = \begin{bmatrix} .5455 & 1.2942 \\ -1.2942 & 17.8494 \end{bmatrix}, \quad Y_2 = \begin{bmatrix} 88.6051 & -3.2240 \\ -3.2240 & 0.5047 \end{bmatrix}. \]

With \( M_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \) and \( N_2 = \begin{bmatrix} -51.5066 & 172.2192 \\ 2.4119 & -12.1811 \end{bmatrix} \), the second local controller is

\[ A_{c2} = \begin{bmatrix} -110.7732 & 323.8752 \\ -26.2210 & 40.9022 \end{bmatrix}, \quad B_{c2} = \begin{bmatrix} 224.4703 \\ 31.6650 \end{bmatrix}, \quad C_{c2} = [258.9 \quad 920.2]. \]

Fig. 4 shows the angle response for the same initial condition. The performance of disturbance attenuation has been greatly improved.
Fig. 3 Angle response for Algorithm 2 (γ = ∞)  
Fig. 4 Angle response for Algorithm 2 (γ = 1)

6 Conclusions
This paper addresses two new kinds of output feedback control design methods with $H_\infty$ performance for fuzzy dynamic systems. The $H_\infty$ fuzzy output feedback controllers can be obtained by using LMI techniques. However, due to using the piecewise differentiable Lyapunov function, one boundary condition has been imposed in the second method, which is the topic of further study.

References
模糊动态系统的 $H_{\infty}$ 输出反馈控制设计
——应用线性矩阵不等式的算法

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摘 要 讨论了一类能用模糊动态系统模型表示的非线性系统的 $H_{\infty}$ 输出反馈控制的设计．
分别基于一个共同的李亚普诺夫方程和一个分段微分的李亚普诺夫方程，给出了两种 $H_{\infty}$
输出反馈模糊控制的新设计方法．通过解一组线性矩阵不等式，可以得到 $H_{\infty}$ 输出反馈控
制器．最后，举例说明了这两种新的设计方法．

关键词 模糊动态系统， $H_{\infty}$ 输出反馈控制，线性矩阵不等式．

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