Stabilization of Discrete-time 2-D T-S Fuzzy Systems Based on New Relaxed Conditions

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Abstract This paper is concerned with the problem of stabilization of the Roesser type discrete-time nonlinear 2-D system that plays an important role in many practical applications. First, a discrete-time 2-D T-S fuzzy model is proposed to represent the underlying nonlinear 2-D system. Second, new quadratic stabilization conditions are proposed by applying relaxed quadratic stabilization technique for 2-D case. Third, for sake of further reducing conservatism, new non-quadratic stabilization conditions are also proposed by applying a new parameter-dependent Lyapunov function, matrix transformation technique, and relaxed technique for the underlying discrete-time 2-D T-S fuzzy system. Finally, a numerical example is provided to illustrate the effectiveness of the proposed results.

Key wordsRoesser model, 2-D discrete systems, Takagi-Sugeno (T-S) fuzzy model, relaxed stabilization conditions

DOI 10.3724/SP.J.1004.2010.00267

In the past three decades, the two-dimensional (2-D) systems have been investigated by many researchers since it could represent a wide range of practical systems, such as those in image data processing and transmission, thermal process, signal filtering, etc. Recently, the 2-D system theory was also frequently used as an analysis tool to some problems, e.g., iterative learning control and repetitive process control. The PI control of discrete linear repetitive processes was investigated in [5]. In [6], the problem of $H_\infty$ control for 2-D discrete state delay systems described by the second Fornasini-Marchesini (FM) state-space model was studied. Due to the application in modeling hybrid systems, $H_\infty$ filtering for 2-D Markovian jump systems was also investigated in [7]. Moreover, stability analysis of 2-D discrete systems described by the FM second model with state saturation was studied in [8]. However, the aforementioned results were only for linear 2-D systems. As well known, most of the actual 2-D systems are nonlinear and the afore-mentioned results do not work in this case. To the best of our knowledge, the corresponding problems on nonlinear 2-D systems have not been fully investigated yet, research in this area should be very important and useful for researchers and designers in this field, which motivates us to carry out this work.

On the other hand, the stability analysis and systematic design of nonlinear systems, with a design model given by the Takagi-Sugeno (T-S) fuzzy model, have been studied by many researchers. Reference [10] has proved that the T-S fuzzy systems can be approximate to any continuous functions in a compact set of $\mathbb{R}^n$ at any precision. This allows the designers to take advantage of conventional linear systems to analyze and design nonlinear systems. Therefore, T-S fuzzy control has become one of the most popular and promising research platform in the model-based fuzzy control, and the theoretic researches on the issue have been conducted actively by many fuzzy control theorists. Among these exiting stabilization conditions for T-S fuzzy systems, most of the works proposed the use of a common quadratic Lyapunov function (CQLF). Other works can be found in dealing with piecewise quadratic Lyapunov functions. Recently, some works also dealt with nonquadratic stability and stabilization with the purpose of further releasing conservatism. As stated in [14], there is still some conservatism to be lifted if we change "something", either the control law or the Lyapunov function or the form of introducing additional variables.

In this paper, the problem of stabilization for Roesser type discrete-time nonlinear 2-D systems will be investigated. A discrete-time 2-D T-S fuzzy model is proposed to represent the underlying discrete-time nonlinear 2-D systems. Based on the attained fuzzy model, new stabilization conditions via CQLF and parallel distributed compensation (PDC) scheme are derived by using relaxed quadratic stabilization techniques. For the sake of further releasing the conservatism, less conservative stabilization conditions are also obtained by using the non-PDC scheme and new relaxed non-quadratic techniques. Unlike the usual 1-D T-S fuzzy systems, the system information is propagated along two independent directions and this fact makes the controller synthesis more complicated, especially for non-quadratic stabilization. Fortunately, this obstacle is overcome by designing appropriate controller gain matrices’ structure. Furthermore, the fact that the relaxed technique provided in this paper was prior to those provided in the existing literature is also proved.

The rest of this paper is organized as Follows. Following the introduction, the discrete-time 2-D T-S fuzzy system is proposed to represent the underlying nonlinear 2-D systems and some important definitions and lemmas are also given in Section 1. In Section 2, new quadratic stabilization conditions are proposed using the PDC scheme and CQLF. To further reduce the conservatism, non-quadratic stabilization conditions are also investigated by using non-PDC scheme and parameter-dependent Lyapunov function (PDLF) in Section 3. In Section 4, an example is given to demonstrate the effectiveness of the results proposed in Sections 2 and 3. Finally, some conclusions are drawn in Section 5.

For simplicity, the notations used are fair standard. For example, $X > 0$ (or $X \geq 0$) means the matrix $X$ is symmetric and positive definite (or symmetric and positive semidefinite). $X^T$ denotes the transpose of $X$. The symbol $I$ represents the identity matrix with an appropriate dimension. " $*$" in a symmetric matrix denotes the transposed element in the symmetric position. For a matrix $P$, $\min(P)$ (or $\max(P)$) means the smallest (largest) eigenvalue of $P$. 

Manuscript received October 6, 2008; accepted March 5, 2009

Supported by National Natural Science Foundation of China (50977008, 60904017, 60774048, 60728307), the Funds for Creative Research Groups of China (60521003), the Program for Cheung Kong Scholars and Innovative Research Team in University (IRT0421), and the 111 Project (B08015), National High Technology Research and Development Program of China (863 Program) (2006AA041243). 1. Key Laboratory of Integrated Automation for the Process Industry, Ministry of Education, Shenyang 110004, P. R. China 2. School of Information Science and Engineering, Northeastern University, Shenyang 110004, P. R. China
1 Problem statement

1.1 Discrete-time 2-D T-S fuzzy model

Consider a class of Roesser type discrete-time nonlinear 2-D systems described by

\[ \mathbf{x}^+(k, l) = \mathbf{z}(k, l) + s(\mathbf{x}(k, l)) \mathbf{u}(k, l) \]  
\[ \mathbf{x}^b(0, l) = \mathbf{f}(l), \quad \mathbf{x}^v(k, 0) = \mathbf{g}(k) \] 

with

\[ \mathbf{x}(k, l) = \begin{bmatrix} \mathbf{x}^b(k, l) \\ \mathbf{x}^v(k, l) \end{bmatrix}, \quad \mathbf{x}^+(k, l) = \begin{bmatrix} \mathbf{x}^b(k + 1, l) \\ \mathbf{x}^v(k, l + 1) \end{bmatrix} \]

where \( \mathbf{x}^b(\cdot) \) is the horizontal state in \( \mathbb{R}^{n_1} \), \( \mathbf{x}^v(\cdot) \) is the vertical state in \( \mathbb{R}^{n_2} \), \( \mathbf{z}() \) is the control input in \( \mathbb{R}^m \), \( z(\cdot) \) and \( s(\cdot) \) are general nonlinear functions satisfying \( z, s \in \mathcal{C}^1 \), \( \mathbf{f}(l) \) and \( \mathbf{g}(k) \) are corresponding boundary conditions along the two independent directions.

By extending the usual 1-D T-S fuzzy modeling method to the 2-D case, a discrete-time 2-D T-S fuzzy model described by the following rules is proposed to represent discrete-time nonlinear 2-D systems (1):

\[
\text{IF } z_1(k, l) = M_{i1}, \ldots, \text{ and } z_L (k, l) = M_{iL}, \text{ THEN,} \\
\mathbf{x}^+(k, l) = A_i \mathbf{x}(k, l) + B_i \mathbf{u}(k, l), \quad i = 1, \ldots, r \\
\mathbf{x}^b(0, l) = \mathbf{f}(l), \quad \mathbf{x}^v(k, 0) = \mathbf{g}(k)
\]

with

\[ A_i = \begin{bmatrix} A_{i1}^{11} & A_{i1}^{12} \\ A_{i2}^{11} & A_{i2}^{12} \end{bmatrix}, \quad B_i = \begin{bmatrix} B_{i1}^{11} \\ B_{i1}^{12} \end{bmatrix} \]

where \( z_p(k, l) \), for \( p = 1, \ldots, L \), are the premise variables, \( M_{ip} \) is the fuzzy set, \( r \) is the number of IF-THEN rules, \( k, l \) are two integers in \( \mathbb{Z}_+ \), and \( A_{i1}^{11}, A_{i2}^{11} \in \mathbb{R}^{n_1 \times n_1}, A_{i1}^{12} \in \mathbb{R}^{n_1 \times n_2}, A_{i2}^{11} \in \mathbb{R}^{n_2 \times n_1}, A_{i2}^{12} \in \mathbb{R}^{n_2 \times n_2}, B_{i1}^{11} \in \mathbb{R}^{n_1 \times m}, B_{i1}^{12} \in \mathbb{R}^{n_2 \times m} \), respectively.

By using product of inference, singleton fuzzifier, and center-average defuzzifier, the overall discrete-time 2-D T-S fuzzy systems can be expressed as follows:

\[ \mathbf{x}^+(k, l) = \sum_{i=1}^{r} h_i(z(k, l)) \left( A_i \mathbf{x}(k, l) + B_i \mathbf{u}(k, l) \right) \] 
\[ \mathbf{x}^b(0, l) = \mathbf{f}(l), \quad \mathbf{x}^v(k, 0) = \mathbf{g}(k) \]

where

\[ h_i(z(k, l)) = \frac{\beta_i(z(k, l))}{\sum_{i=1}^{r} \beta_i(z(k, l))}, \beta_i(z(k, l)) = \Pi_{k=1}^{z_i} M_{k_i}(z(k, l)) \]

In this paper, for a matrix \( X \), the following notations will be adopted for simplicity:

\[ h_i = h_i(z(k, l)), X_z = \sum_{i=1}^{r} h_i X_i, X_z^{-1} = \left( \sum_{i=1}^{r} h_i X_i \right)^{-1} \]

**Remark 1.** Based on the discrete-time 2-D T-S fuzzy model (4), the problem of controller synthesis for systems (1) could be implemented under the framework for linear 2-D systems. However, it is worth noting that membership functions (MFs) play an important part in system (4), hence how to make good use of the information of them in the process of controller synthesis seems meaningful and interesting\[^{18}\]. Furthermore, the underlying 2-D T-S system’s information is propagated along two independent directions and this fact makes the problem of stabilization more complicated, especially for the case of non-quadradic stabilization.

1.2 Definition and lemma

Denote \( X_r = \sup \{|\mathbf{x}(k, l)| : r = k + l\} \), and we firstly give the definition of asymptotical stability for system (4).

**Definition 1.** The discrete-time 2-D T-S fuzzy system (4) is asymptotically stable if \( \lim_{k \to \infty} X_r = 0 \) with the initial and boundary conditions (2).

We end this section with an useful lemma that will play an important role in the derivation of one of our results.

**Lemma 1.** For two symmetric matrices \( P > 0 \) and \( Q > 0 \), the inequality \( A^T P A - Q < 0 \) holds, if there exist a matrix \( G \) such that

\[ \begin{bmatrix} P & -Q \\ G A & G + C^T - Q \end{bmatrix} > 0. \]

2 Stabilization conditions via PDC scheme and CQLF

In this section, new quadratic stabilization conditions for system (4) via PDC scheme and CQLF will be proposed by using some relaxed quadratic stabilization techniques.

With the idea of extending the so-called PDC scheme for usual 1-D T-S fuzzy systems to the 2-D case, we use the controller structure incorporating a set of fuzzy rules expressed as

\[ \mathbf{u}(k, l) = \sum_{i=1}^{r} h_i K_i \mathbf{x}(k, l) = K \mathbf{x}(k, l) \]

Then, the closed-loop system of (4) and (6) is shown as follows:

\[ \mathbf{x}^+(k, l) = \sum_{i=1}^{r} \sum_{j=1}^{r} h_i h_j (A_i + B_i K_j) \mathbf{x}(k, l) = (A_x + B_x K) \mathbf{x}(k, l) \]

Hence, the problem which we are dealing with now is how to design the gain matrix of \( K \) that stabilizes the 2-D closed-loop system (7) with less conservative quadratic stabilization conditions.

**Theorem 1.** The discrete-time 2-D T-S fuzzy systems (4) is asymptotically stable via the controller (6) if there exists appropriately dimensional matrices \( X_1 > 0, X_2 > 0, F_i, Q_{ii}, \) and \( Q_{ij} = Q_{ji}^T \), with

\[ F_i = \begin{bmatrix} F_{i1} \\ F_{i2} \end{bmatrix}, \quad Q_{ii} = \begin{bmatrix} Q_{i1}^{11} & Q_{i1}^{12} & Q_{i1}^{13} & Q_{i1}^{14} \\ Q_{i2}^{11} & Q_{i2}^{12} & Q_{i2}^{13} & Q_{i2}^{14} \\ Q_{i3}^{11} & Q_{i3}^{12} & Q_{i3}^{13} & Q_{i3}^{14} \\ Q_{i4}^{11} & Q_{i4}^{12} & Q_{i4}^{13} & Q_{i4}^{14} \end{bmatrix} \]

\[ Q_{ij} = \begin{bmatrix} Q_{i1}^{21} & Q_{i1}^{22} & Q_{i1}^{23} & Q_{i1}^{24} \\ Q_{i2}^{21} & Q_{i2}^{22} & Q_{i2}^{23} & Q_{i2}^{24} \\ Q_{i3}^{21} & Q_{i3}^{22} & Q_{i3}^{23} & Q_{i3}^{24} \\ Q_{i4}^{21} & Q_{i4}^{22} & Q_{i4}^{23} & Q_{i4}^{24} \end{bmatrix}, \quad Q_{ji} = Q_{ij}^T \]
such that the following LMIs hold:

$$\Theta_{ij} \leq Q_{ii}, i = 1, \ldots, r$$  \hspace{1cm} (8)$$

$$\Theta_{ij} + \Theta_{ji} \leq Q_{ij} + Q_{ji}, \, i \neq j, \, i, j = 1, \ldots, r$$  \hspace{1cm} (9)$$

where

$$
\begin{bmatrix}
Q_{11} & Q_{12} & \cdots & Q_{1r} \\
Q_{21} & Q_{22} & \cdots & Q_{2r} \\
\vdots & \vdots & \ddots & \vdots \\
Q_{r1} & Q_{r2} & \cdots & Q_{rr}
\end{bmatrix}
\leq 0
$$

where, for $i, j = 1, \ldots, r$, we have

$$\Theta_{ij} = 
\begin{bmatrix}
-X_1 & 0 & \Theta_{ij}^{13} & \Theta_{ij}^{14} \\
* & -X_2 & \Theta_{ij}^{23} & \Theta_{ij}^{24} \\
* & * & -X_1 & 0 \\
* & * & * & -X_2
\end{bmatrix}
$$

$$\Theta_{ij}^{13} = (A_1^T X_1 + B_1^T F_j^T)^T, \Theta_{ij}^{14} = (A_1^T X_1 + B_1^T F_j)^T$$

$$\Theta_{ij}^{23} = (A_2^T X_2 + B_1^T F_j^T)^T, \Theta_{ij}^{24} = (A_2^T X_2 + B_1^T F_j^T)^T$$

Moreover, the controller gain matrices could be given by

$$K_k = [F_1 X_{k,1}^T \cdots F_r X_{k,r}^T].$$

**Proof.** Consider a CQLF given as follows:

$$V(x(k,l)) = x^T(k,l) P x(k,l)$$  \hspace{1cm} (11)$$

where $P$ is a positive definite matrix of the following form:

$$P = \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix}$$

where $P_1 \in \mathbb{R}^{n_1 \times n_1}$ and $P_2 \in \mathbb{R}^{n_2 \times n_2}$. The variation of (11) is given by

$$\Delta V(x(k,l)) = x^T(k,l) P x(k,l) - x^T(k,l) P x(k,l) = x^T(k,l) [(A_1 + B_1 K_k)^T P (A_1 + B_1 K_k) - P] x(k,l)$$  \hspace{1cm} (12)$$

Then, it is easy to see that system (4) is asymptotically stable if we have

$$(A_1 + B_1 K_k)^T P (A_1 + B_1 K_k) - P < 0$$  \hspace{1cm} (13)$$

Using the Schur complement lemma, (13) is equivalent to the following inequality:

$$\begin{bmatrix}
-P \\
* \\
(A_1 + B_1 K_k)^T P \\
* \\
-P
\end{bmatrix} < 0$$  \hspace{1cm} (14)$$

Pre- and post-multiplying both sides of (14) by $\text{diag}(P^{-1}, P^{-1})$ and applying the change of variables $X = P^{-1} F_i X$ leads to

$$\sum_{i=1}^{r} \sum_{j=1}^{r} h_i h_j \Theta_{ij} = \begin{bmatrix}
-X_1 \\
* \\
(A_1 X_1 + B_1 F_i)^T \\
* \\
-X_2
\end{bmatrix} < 0$$  \hspace{1cm} (15)$$

where $X = \begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix}$ and $\Theta_{ij}$ are defined in (8) and (9).

Reordering the expression of $\sum_{i=1}^{r} \sum_{j=1}^{r} h_i h_j \Theta_{ij}$ and using (8) and (9), we can obtain

$$\sum_{i=1}^{r} \sum_{j=1}^{r} h_i h_j \Theta_{ij} = \sum_{i=1}^{r} h_i^2 \Theta_{ii} + \sum_{i=1}^{r} \sum_{j>i} h_i h_j (\Theta_{ij} + \Theta_{ji}) \leq \sum_{i=1}^{r} h_i^2 \Theta_{ii} + \sum_{i=1}^{r} \sum_{j>i} h_i h_j (Q_{ij} + Q_{ji})$$

Thus, if (10) holds, $\sum_{i=1}^{r} \sum_{j=1}^{r} h_i h_j \Theta_{ij} < 0$ evidently holds. In other words, the discrete-time 2-D system (4) is asymptotically stable via the fuzzy controller (6).

**Remark 2.** By extending the relaxed quadratic stabilization technique for 1-D T-S fuzzy system[12] to the 2-D case and modifying somewhat in view of adapting to the 2-D setting, new quadratic stabilization conditions in Theorem 1 are less conservative than those only using common quadratic stabilization methods. However, in the process of the derivation, the nonlinear functions used to blend the linear models are not involved in the LMI conditions. Thus, these conditions remain conservative, and how to further reduce the conservatism seems meaningful and interesting.

### 3 Stabilization conditions via non-PDC scheme and PDLF

As well known, MFs play important roles in the T-S fuzzy systems. It has a chance to further reduce the conservatism if we consider information of MFs in the process of controller design. To further release the conservatism, new stabilization conditions for system (4) will be proposed by using non-PDC scheme, PDLF, and new relaxed techniques in this section. Here, the non-quadratic control law is designed as

$$u(k,l) = \left( \sum_{i=1}^{r} h_i F_i \right) \left( \sum_{i=1}^{r} h_i G_i \right)^{-1} x(k,l) \times F_i G_i^{-1} x(k,l)$$  \hspace{1cm} (17)$$

where $F_i$ and $G_i$ are appropriately dimensional matrices to be determined and have the following matrix structures:

$$F_i = \begin{bmatrix} F_i^1 & F_i^2 \end{bmatrix}, \quad G_i = \begin{bmatrix} G_i^1 & 0 \\ 0 & G_i^2 \end{bmatrix}$$  \hspace{1cm} (18)$$

**Theorem 2.** The discrete-time 2-D T-S fuzzy system (4) with the non-quadratic controller (17) is asymptotically stable if there exists appropriately dimensional ma-
traces $P_t > 0$, $F_t, G_t, R_{t_l}^{mn}, u_t^{mn} = (R_t^{mn})^T$, with

$$P_t = \begin{bmatrix} P_t^1 & 0 \\ 0 & P_t^2 \end{bmatrix}, P_t^1 \in \mathbb{R}^{n_1 \times n_1}, P_t^2 \in \mathbb{R}^{n_2 \times n_2}$$

$$R_{t_l}^{mn} = \begin{bmatrix} R_{t_l}^{mn}(11) & \cdots & R_{t_l}^{mn}(14) \\ \vdots & \ddots & \vdots \\ * & \cdots & * \end{bmatrix} R_{t_l}^{mn}(44)$$

such that the following LMI holds:

$$\begin{aligned}
\mathcal{T}_{t_l}^{mn} & > R_{t_l}^{mn}, i, m, n = 1, \ldots, r \\
\mathcal{T}_{m}^{mn} & > R_{m}^{mn} + \mathcal{T}_{m}^{mn} + R_{m}^{mn}, i \neq j, i, m, n = 1, \ldots, r
\end{aligned}$$

(19)

(20)

where, for $i, j, m, n = 1, \ldots, r$, we have

$$Y_{t_l}^{mn} = \begin{bmatrix} P_t^1 & 0 & \mathcal{T}_{t_l}^{mn}(1, 3) & \mathcal{T}_{t_l}^{mn}(1, 4) \\ * & P_t^2 & \mathcal{T}_{t_l}^{mn}(2, 3) & \mathcal{T}_{t_l}^{mn}(2, 4) \\ * & * & \mathcal{T}_{t_l}^{mn}(3, 3) & 0 \\ * & * & * & \mathcal{T}_{t_l}^{mn}(4, 4) \end{bmatrix}$$

(21)

Proof. Consider a new non-quadratic Lyapunov function for discrete-time 2-D T-S system as follows:

$$V(x(k, l)) = x^T(k, l)G_x z^T P_x G_x^T x(k, l)$$

(22)

First, let us check the existence of $G_x^{-1}$. Note that if these conditions of Theorem 2 hold true, we have with inequalities (19): $G_{t_l} + (G_t^{mn})^T - P_{t_l} > 0$ (i = 1, \ldots, r) and $G_{t_l} + (G_t^{mn})^T - P_{t_l} > 0$ (i = 1, \ldots, r). Therefore, $\sum_i h_i(G_t + G_t^{mn} - P_t) > 0$ (i = 1, \ldots, r), which ensures that $G_x^{-1}$ exists.

Second, we check the non-quadratic Lyapunov function (22)’s validity. We can write

$$\overline{\mathcal{L}}_{t_l}^m \left| x(k, l) \right|^2 \leq V \leq \overline{\mathcal{L}}_{t_l}^m \left| x(k, l) \right|^2 \overline{G_x z^T G_x^{-1} x(k, l)}$$

(23)

where $\overline{\mathcal{L}} = \min_i (P_t^1)$ and $\overline{\mathcal{L}} = \max_i (P_t^2)$.

As $(G_x^{-1} G_x^T)^{-1} = G_x G_x^T$ and with $\mu = \min_i (G_x G_x^T)$ and $\overline{\mu} = \max_i (G_x G_x^T)$, (23) becomes $\overline{\mathcal{L}}_{t_l}^m \left| x(k, l) \right|^2 \leq V \leq \overline{\mathcal{L}}_{t_l}^m \left| x(k, l) \right|^2 \overline{G_x z^T G_x^{-1} x(k, l)}$, which ensures $V(x(k, l))$ to be a candidate Lyapunov function.

Then, its variation is written as

$$\Delta V(x(k, l)) = x^T(k, l)((A_x + B_x F_x G_x^{-1})^T G_x + P_x G_x^{-1} - G_x P_x G_x^{-1}) x(k, l)$$

(24)

where

$$G_x = \begin{bmatrix} \sum_{i=1}^r h_i(z(k + 1, l)) G_i^{11} & 0 \\ 0 & \sum_{i=1}^r h_i(z(k + 1, l)) G_i^{22} \\ \sum_{i=1}^r h_i(z(k + 1, l)) P_i^1 & 0 \\ 0 & \sum_{i=1}^r h_i(z(k + 1, l)) P_i^2 \end{bmatrix}$$

and

$$G_x = \begin{bmatrix} \sum_{i=1}^r h_i(z(k + 1, l)) G_i^{11} & 0 \\ 0 & \sum_{i=1}^r h_i(z(k + 1, l)) G_i^{22} \\ \sum_{i=1}^r h_i(z(k + 1, l)) P_i^1 & 0 \\ 0 & \sum_{i=1}^r h_i(z(k + 1, l)) P_i^2 \end{bmatrix}$$

(25)

(26)

Here, $h_i(z(k + 1, l))$ and $h_i(z(k + 1, l + 1))$ are two different one-step ahead MFs produced by the fact that the 2-D systems’ information is propagated along the two independent directions. Therefore, in the derivation of the relaxed non-quadratic stabilization conditions, we should consider this difference via solving more LMIs as a tradeoff.

Multiplying the left side of (24) by $G_x^T$ and the right by $G_x$, we can easily verify that system (4) with the non-quadratic controller (17) is asymptotically stable if we have the following inequality:

$$(G_x^T A_x^T + F_x^T B_x) G_x + G_x^T P_x G_x^T (A_x G_x + B_x F_x) - P_x < 0$$

(26)

Using Lemma 1 with $A = G_x^T (A_x G_x - B_x F_x)$ leads to

$$\begin{bmatrix} \sum_{m=1}^r \sum_{n=1}^r h_m(z(k + 1, l)) h_n(z(k + 1, l + 1)) \\ \sum_{i=1}^r h_i^2 \mathcal{T}_{i}^{mn} + \sum_{i=1}^{r-1} \sum_{j>i} h_i h_j (\mathcal{T}_{i}^{mn} + \mathcal{T}_{j}^{mn}) \end{bmatrix} > 0$$

(26)

where $\mathcal{T}_{i}^{mn}$, $\mathcal{T}_{j}^{mn}$, and $\mathcal{T}_{i}^{mn}$ are defined in (19) and (20). On the other hand, noting (19) and (20) and applying the Schur complement lemma, we can obtain

$$\begin{bmatrix} \sum_{m=1}^r \sum_{n=1}^r h_m(z(k + 1, l)) h_n(z(k + 1, l + 1)) \\ \sum_{i=1}^r h_i^2 \mathcal{T}_{i}^{mn} + \sum_{i=1}^{r-1} \sum_{j>i} h_i h_j (\mathcal{T}_{i}^{mn} + \mathcal{T}_{j}^{mn}) \end{bmatrix} > 0$$

(26)

where $\eta^T = [h_1 I \ h_2 I \ \cdots \ h_r I]$ and $R_t^{mn}$ is defined in (21).

Thus, if (21) holds, (25) evidently holds. In other words, the discrete-time 2-D system (4) is asymptotically stable via the fuzzy controller (17).

Remark 3. Unlike the usual 1-D T-S fuzzy systems, the key feature of a 2-D T-S fuzzy systems is that the system information is propagated along the two independent directions. For the underlying Roesser type 2-D T-S fuzzy...
system (4) which could model a wide range of practical systems, two different one-step ahead MFs \(h_i(z(k+1, l))\) and \(h_j(z(k, l+1))\) for horizontal and vertical directions, respectively) are produced for conceiving relaxed non-quadratic stabilization conditions. Furthermore, due to the structure of system matrices \(A_i\) and \(B_i\) in (3), those matrices \(\Sigma_{ij}^m\) in Theorem 2 also represent the information along the two independent directions. These two facts lead to more LMIs and complicated matrices structures than the usual 1-D T-S fuzzy systems. This fact will also be illustrated in Section 4.

In [15], the authors proposed a novel form of introducing additional variables to usual 1-D T-S fuzzy systems. The following corollary will be attained if we extend their technique to the 2-D setting.

**Corollary 1.** The discrete-time 2-D T-S fuzzy system (4) with the non-quadratic controller (17) is asymptotically stable if there exists appropriately dimensional matrices \(F_i > 0, F_r, G_i, \Sigma_{ij}^m (i \neq j)\) and symmetric matrices \(Q_{ii}^m\), with

\[
P_i = \begin{bmatrix}
P_i^1 & 0 \\
0 & P_i^2
\end{bmatrix}, 
P_i^1 \in \mathbb{R}^{n_1 \times n_1}, P_i^2 \in \mathbb{R}^{n_2 \times n_2}
\]

\[
Q_{ii}^m = \begin{bmatrix}
Q_{ii}^m(11) & \cdots & Q_{ii}^m(14) \\
\vdots & \ddots & \vdots \\
Q_{ii}^m(31) & \cdots & Q_{ii}^m(44)
\end{bmatrix}
\]

\[
Q_{ij}^m = \begin{bmatrix}
Q_{ij}^m(11) & Q_{ij}^m(12) & Q_{ij}^m(13) & Q_{ij}^m(14) \\
Q_{ij}^m(21) & Q_{ij}^m(22) & Q_{ij}^m(23) & Q_{ij}^m(24) \\
Q_{ij}^m(31) & Q_{ij}^m(32) & Q_{ij}^m(33) & Q_{ij}^m(34) \\
Q_{ij}^m(41) & Q_{ij}^m(42) & Q_{ij}^m(43) & Q_{ij}^m(44)
\end{bmatrix}
\]

such that the following LMIs hold:

\[
\Sigma_{ii}^m > Q_{ii}^m, \quad i, m, n = 1, \ldots, r
\]

\[
\Sigma_{ij}^m + (\Sigma_{ij}^m)^T > Q_{ij}^m + (Q_{ij}^m)^T
\]

\[
2Q_{ii}^m + 2Q_{ij}^m + (Q_{ij}^m)^T > 0
\]

with (28) \(\sim\) (29) and the fact \(W + W^T > 0\) is equivalent to \(W > 0\) for a real matrix, we have

\[
\sum_{i=1}^{r} h_i^2 \Sigma_{ii}^m + \sum_{i=1}^{r-1} \sum_{j \neq i} h_i h_j \Sigma_{ij}^m
\]

with \(r \geq 2\) and the fact \(W + W^T > 0\) is equivalent to \(W > 0\) for a real matrix, we have

\[
\sum_{i=1}^{r} h_i^2 Q_{ii}^m + \sum_{i=1}^{r-1} \sum_{j \neq i} h_i h_j Q_{ij}^m
\]

Similar to the proof of Theorem 2, (30) guarantees

\[
P_i \overset{+}{\sim} \sum_{i=1}^{r} h_i G_i + B_i F_i + G_{z+} + G_{z+}^T - P_z + \Sigma_{ij}^m (i \neq j) > 0.
\]

Hence, the discrete-time 2-D T-S fuzzy system (4) with the non-quadratic controller (17) is asymptotically stable.

**Remark 4.** The number of additional variables \(Q_{ij}^m\) (Corollary 1) is \(r^4\) while the number of additional variables \(R_{ij}^l (i \leq j)\) (Theorem 2) is \(|r^2 + r|/2\). It is easy to see that \(|r^2 + r|/2 < r^4\) for all \(r \geq 2\), i.e., Theorem 2 requires less computation. Moreover, the fact that stabilization conditions derived by Theorem 2 are class are more conservative will be proved in the following proposition.

**Proposition 1.** Stabilization conditions (19) \(\sim\) (21) hold, if stabilization conditions (28) \(\sim\) (30) hold.

**Proof.** Recalling the stabilization conditions (28) and (29) proposed in Corollary 1, we have

\[
\Sigma_{ii}^m > Q_{ii}^m, \quad i, k, l = 1, \ldots, r
\]

\[
\Sigma_{ij}^m + (\Sigma_{ij}^m)^T > Q_{ij}^m + (Q_{ij}^m)^T
\]

\[
2Q_{ii}^m + 2Q_{ij}^m + (Q_{ij}^m)^T > 0
\]

where \(\Sigma_{ij}^m (i, j, m, n = 1, \ldots, r)\) have the same definitions as in Theorem 2.

**Proof.** From the proof of Theorem 2, it is easy to see that the discrete-time 2-D T-S fuzzy system (4) with the non-quadratic controller (17) is asymptotically stable if the inequality (26) holds.

Reordering the expression of the term

\[
P e = \begin{bmatrix}
P e & 0 \\
0 & P e
\end{bmatrix}, 
P e \in \mathbb{R}^{2 \times 2}
\]

\[
\begin{bmatrix}
A_i G_i & B_i F_i & G_{z+} + G_{z+}^T - P_z + \Sigma_{ij}^m (i \neq j)
\end{bmatrix}
\]

\[
= \begin{bmatrix}
P e & 0 \\
0 & P e
\end{bmatrix}, 
P e \in \mathbb{R}^{2 \times 2}
\]

\[
\sum_{i=1}^{r} h_i^2 \Sigma_{ii}^m + \sum_{i=1}^{r-1} \sum_{j \neq i} h_i h_j \Sigma_{ij}^m
\]

\[
\sum_{i=1}^{r} h_i^2 Q_{ii}^m + \sum_{i=1}^{r-1} \sum_{j \neq i} h_i h_j Q_{ij}^m
\]

with (28) \(\sim\) (29) and the fact \(W + W^T > 0\) is equivalent to \(W > 0\) for a real matrix, we have

\[
\sum_{i=1}^{r} h_i^2 \Sigma_{ii}^m + \sum_{i=1}^{r-1} \sum_{j \neq i} h_i h_j \Sigma_{ij}^m
\]

\[
\sum_{i=1}^{r} h_i^2 Q_{ii}^m + \sum_{i=1}^{r-1} \sum_{j \neq i} h_i h_j Q_{ij}^m
\]

4 Numerical example

**Example.** Consider the following nonlinear differential equation:

\[
\frac{\partial q(x, t)}{\partial x} = a_1 q(x, t) + a_2 q^2(x, t) + a_3 \sin^2(q(x, t)) + b f(x, t)
\]

where the initial and boundary conditions \(q(x, 0) = q_0(x)\) and \(q(0, t) = q_0(t), q(x, t)\) is the variable function, \(a_0, a_1, a_2, b\) are real coefficients, and \(f(x, t)\) is the input function.

Next, we will establish the state space model for the above-mentioned nonlinear differential equation. Let us define

\[
x^h_i (x, t) = \frac{\partial q(x, t)}{\partial t} = a_2 q(x, t)
\]

\[
x_i^e (x, t) = q(x, t)
\]
Then, the following 2-D state space model can be easily obtained:

\[
\begin{bmatrix}
\frac{\partial x^h_c(x, t)}{\partial x} \\
\frac{\partial x^f_c(x, t)}{\partial t}
\end{bmatrix} = \begin{bmatrix}
a_1 & a_1a_2 + a_0 \sin^2(x^c(x, t)) \\
1 & a_2
\end{bmatrix} \times
\begin{bmatrix}
x^h_c(x, t) \\
x^f_c(x, t)
\end{bmatrix} + \begin{bmatrix}
b \\
0
\end{bmatrix} u_c(x, t)
\]

with boundary conditions: \( x^h_c(0, t) = q_1(t) - a_2q_2(t) \) and \( x^f_c(x, t) = q_1(x) \).

To obtain a 2-D T-S fuzzy representation for this 2-D nonlinear system, consider the following rules obtained for \( \sin^2(x^c(x, t)) \):

**IF** \( \sin^2(x^c(x, t)) = \sin^2(x^c(x, t)) \), THEN

\[
\begin{bmatrix}
\frac{\partial x^h_c(x, t)}{\partial x} \\
\frac{\partial x^f_c(x, t)}{\partial t}
\end{bmatrix} = A_1^c \begin{bmatrix}
x^h_c(x, t) \\
x^f_c(x, t)
\end{bmatrix} + B_1^c u_c(x, t)
\]

**IF** \( \sin^2(x^c(x, t)) = \sin^2(x^c(x, t)) \), THEN

\[
\begin{bmatrix}
\frac{\partial x^h_c(x, t)}{\partial x} \\
\frac{\partial x^f_c(x, t)}{\partial t}
\end{bmatrix} = A_2^c \begin{bmatrix}
x^h_c(x, t) \\
x^f_c(x, t)
\end{bmatrix} + B_2^c u_c(x, t)
\]

where \( A_1^c = \begin{bmatrix} a_1 & a_1a_2 + a_0 \\ 1 & a_2 \end{bmatrix}, B_1^c = \begin{bmatrix} b \\ 0 \end{bmatrix}, A_2^c = \begin{bmatrix} a_1 & a_1a_2 + a_0 \\ 1 & a_2 \end{bmatrix}, \) and \( B_2^c = B_1^c \).

Without loss of generality, we choose the MFs: \( h_1(x, t) = 1 - \sin^2(x^c(x, t)) \) and \( h_2(x, t) = \sin^2(x^c(x, t)) \).

The above-mentioned 2-D T-S fuzzy systems are discretized with sampling times \( T_1 \) and \( T_2 \) corresponding to variables \( x \) and \( t \), respectively. The obtained discrete-time 2-D fuzzy systems are given by

**IF** \( \sin^2(x^c(k, l)) = \sin^2(x^c(k, l)) \), THEN

\[
\begin{bmatrix}
x^h(k + 1, l) \\
x^f(k, l + 1)
\end{bmatrix} = A_1 \begin{bmatrix}
x^h(k, l) \\
x^f(k, l)
\end{bmatrix} + B_1 u(k, l)
\]

**IF** \( \sin^2(x^c(k, l)) = \sin^2(x^c(k, l)) \), THEN

\[
\begin{bmatrix}
x^h(k + 1, l) \\
x^f(k, l + 1)
\end{bmatrix} = A_2 \begin{bmatrix}
x^h(k, l) \\
x^f(k, l)
\end{bmatrix} + B_2 u(k, l)
\]

where \( A_1 = \begin{bmatrix} 1 + a_1T_1 \\ T_2 \end{bmatrix} \begin{bmatrix} a_1a_2T_1 \\ T_2 \end{bmatrix}, B_1 = \begin{bmatrix} bT_1 \\ 0 \end{bmatrix} \),

\[
A_2 = \begin{bmatrix} 1 + a_1T_1 \\ T_2 \end{bmatrix} \begin{bmatrix} a_1a_2 + a_0T_1 \\ T_2 \end{bmatrix}, \) and \( B_2 = B_1 \).
5 Conclusion

This paper has presented a solution to the problem of stabilizing the Roesser type discrete-time nonlinear 2-D system. The underlying nonlinear system is represented by a discrete-time 2-D T-S fuzzy model, and then two kinds of stabilization conditions are derived by using new relaxed techniques, respectively. A numerical example shows the effectiveness of the proposed results.

References