Non-fragile $H_\infty$ Filtering for Discrete-time Systems with Finite Word Length Consideration

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Abstract The nonfragile $H_\infty$ filtering problem affected by finite word length (FWL) for linear discrete-time systems is investigated in this paper. The filter to be designed is assumed to be with additive gain variations, which reflect the FWL effects on filter implementation. A notion of structured vertex separator is proposed to deal with the problem and exploited to develop sufficient conditions for the nonfragile $H_\infty$ filter design in terms of a set of linear matrix inequalities (LMIs). The design renders the augmented system asymptotically stable and guarantees the $H_\infty$ attenuation level less than a prescribed level. A numerical example is given to illustrate the effect of the proposed method.

Key words Linear system, $H_\infty$ filter, fragility, finite word length (FWL), linear matrix inequality (LMI)

The optimal $H_\infty$ filtering problem for linear discrete-time systems has been attracting considerable attention for researchers for the past decades[1]. To reduce the design conservativeness, there have been many works providing a decoupling between the Lyapunov matrix and the system dynamic matrix by introducing slack variables[2–3].

All the above works are based on an implicit assumption that the filter will be implemented exactly. However, in the course of filter implementation with different design algorithms, it turns out that the filters can be sensitive with respect to errors in the filter coefficients. The sources for this include, but not limited to, imprecision in analogue-digital conversion, fixed word length, finite resolution instrumentation, and numerical roundoff errors. By means of several examples, it is demonstrated in the control design formalism[4] that relatively small perturbations in controller parameters could even destabilize the closed-loop system. So a significant issue is how to design a filter or controller for a given plant, such that the filter or controller is insensitive to some amount of error with respect to its gain, i.e., the designed filter or controller is resilient or nonfragile. This issue has received some attention from the control systems community. Recently, the problem of resilient Kalman filtering with respect to estimator gain perturbations was considered in [5]. In [6], the problem of designing robust resilient linear filtering for a class of continuous-time systems with norm-bounded uncertainty was investigated.

Note that the works discussed above deal with the nonfragile problem with the consideration of norm-bounded type of gain uncertainty. However, this type of uncertainty cannot reflect the uncertain information due to the finite word length (FWL) effects exactly. Correspondingly, the interval type of uncertainty[7] is more exact than the former type to describe the uncertain information, but till now, there is no work on the nonfragile filter design problem that takes account of interval gain uncertainty. On the other hand, the vertices of the set of uncertain parameters grow exponentially with the number of uncertain parameters, which may result in numerical problems for systems with high dimensions. These problems motivate our work in this paper.

This paper is concerned with the problem of nonfragile $H_\infty$ filter design for linear time-invariant (LTI) discrete-time systems with FWL consideration. The filter to be designed is assumed to be with the additive gain variations of interval type, which are due to the FWL effects when the filter is implemented. First, an LMI-based sufficient condition is given for the solvability of the nonfragile $H_\infty$ filtering problem, which requires checking all of the vertices of the set of uncertain parameters that grow exponentially with the number of uncertain parameters. It will be very difficult to apply the result to the systems with high orders. To overcome the difficulty, a notion of structured vertex separator is proposed to deal with the problem, and exploited to develop sufficient conditions for the nonfragile $H_\infty$ filter design in terms of solutions to a set of LMIs. The structured vertex separator-based method can significantly reduce the number of the LMI constraints involved in the design condition. Moreover, we adopt the slack variable method[9] to realize the decoupling between the Lyapunov matrix and the system dynamic matrix, which reduces the design conservativeness. The designs guarantee the asymptotic stability of the estimation errors, and the $H_\infty$ performance of the system from the exogenous signals to the estimation errors less than a prescribed level. On the other hand, the existing method given in [5–6] and [8], for the nonfragile problem with norm-bounded gain variations, is also applicable to the nonfragile $H_\infty$ filtering problem considered here. But this method is more conservative than our newly proposed one, which will be shown in Section 2.

Notation. For a matrix $E$, $E^T$ and $E^{-1}$ denote its transpose and inverse, respectively, if they exist. For a column-rank deficient matrix $H$, $N_H$ denotes a matrix whose columns form a basis for the null space of $H$. $I$ denotes the identity matrix with an appropriate dimension. $0_{i\times j}$ represents zero matrix of $i$ rows and $j$ columns. The symbol $*$ within a matrix represents the symmetric entries.

1 Problem statement and preliminaries

1.1 Problem statement

Consider an LTI discrete-time model described by

$$x(k+1) = Ax(k) + B_1u(k)$$
$$y(k) = C_1x(k) + D_2w(k)$$

(1)

where $x(k) \in \mathbb{R}^n$ is the state, $y(k) \in \mathbb{R}^m$ is the measured output, $\omega(k) \in \mathbb{R}^p$ is the disturbance input, and $z(k) \in \mathbb{R}^q$ is the regulated output. $A, B_1, C_1, C_2$, and $D_2$ are known constant matrices of appropriate dimensions.
To cope with the filtering problem, we consider a discrete-time filter with gain variations of the following form

\[
\begin{align*}
\xi(k+1) &= (A_F + \Delta A_F)\xi(k) + (B_F + \Delta B_F)g(k) \\
z_F(k) &= (C_F + \Delta C_F)\xi(k)
\end{align*}
\] (2)

where \(\xi(k)\in \mathbb{R}^n\) is the filter state, \(z_F(k)\) is the estimation of \(z(k)\), and the constant matrices \(A_F, B_F, C_F\) are filter matrices to be designed, \(\Delta A_F, \Delta B_F, \Delta C_F\) represent the interval type of additive gain variations with the following forms

\[
\begin{align*}
\Delta A_F &= [\delta_{ai,j}]_{n \times n}, [\delta_{ai,j}] \leq \delta_a, i, j = 1, \cdots, n \\
\Delta B_F &= [\delta_{hi,j}]_{n \times p}, [\delta_{hi,j}] \leq \delta_b, i = 1, \cdots, n, j = 1, \cdots, p \\
\Delta C_F &= [\delta_{ij}]_{p \times n}, [\delta_{ij}] \leq \delta_c, i = 1, \cdots, p, j = 1, \cdots, n
\end{align*}
\] (3)

Remark 1. The additive gain variation model of form (3) is from [7], which has been extensively used to describe the FWL effects.

Let \(\mathbf{e}_1 \in \mathbb{R}^n, \mathbf{h}_i \in \mathbb{R}^p\), and \(\mathbf{g}_j \in \mathbb{R}^q\) denote the column vectors, in which the \(k\)-th elements are 1 and the others equal 0. Then, the gain variations of the form (3) can be described as

\[
\begin{align*}
\Delta A_F &= \sum_{i=1}^{n}\sum_{j=1}^{n} \delta_{ai,j}\mathbf{e}_i\mathbf{e}_j^T, \quad \Delta B_F = \sum_{i=1}^{n}\sum_{j=1}^{p} \delta_{hi,j}\mathbf{h}_i\mathbf{h}_j^T \\
\Delta C_F &= \sum_{i=1}^{p}\sum_{j=1}^{n} \delta_{ij}\mathbf{g}_i\mathbf{g}_j^T
\end{align*}
\]

Combining filter (2) with system (1), we obtain the filter error system as

\[
\begin{align*}
\mathbf{z}_e(k+1) &= A_e\mathbf{z}_e(k) + B_\omega \mathbf{w}(k) \\
\mathbf{z}_e(k) &= C_e\mathbf{z}_e(k)
\end{align*}
\] (4)

where \(\mathbf{z}_e(k) = [\mathbf{z}(k)^T, \mathbf{z}(k)^T]^T\) and \(\mathbf{z}_e(k) = \mathbf{z}(k) - z_F(k)\) is the estimation error, and

\[
A_e = \begin{bmatrix} A & 0 \\ (B_F + \Delta B_F)C_2 & A_F + \Delta A_F \end{bmatrix}
\]

\[
B_e = \begin{bmatrix} B_1 \\ (B_F + \Delta B_F)D_{21} \end{bmatrix}, \quad C_e = [C_1 - C_F - \Delta C_F]
\]

The transfer function matrix of the augmented system (4) from \(\mathbf{w}\) to \(\mathbf{z}_e\) is given by

\[
G_{z_e,\omega}(z) = C_e(zI - A_e)^{-1}B_e
\]

Then, the problem under consideration in this paper is as follows.

**Non-fragile \(H_\infty\) filtering problem with filter gain variations:** Given a positive constant \(\gamma\), find a filter described by (2) with the gain variations of the form (3), such that the resulting system (4) is asymptotically stable and \([G_{z_e,\omega}(z)] < \gamma\).

### 1.2 Useful lemmas

**Lemma 1** [9]. Let matrices \(Q = Q^T, G, \) and a compact subset of real matrices \(H\) be given. Then, the following statements are equivalent:

1. For each \(H \in H\)
   \[
   \xi^TQ\xi < 0, \quad \text{for all } \xi \neq 0, \text{ such that } H\xi = 0
   \]
2. There exists \(\Theta = \Theta^T\) such that
   \[
   Q + G^T\Theta G < 0, \quad \text{and } N_H^T\Theta N_H \geq 0 \text{ for all } H \in H
   \]

**Lemma 2** [10]. Let \(G_{\omega,\omega}(z) = C_0(zI - A_0)^{-1}B_0\); then, \(A_0\) is Shur stable, and \([G_{\omega,\omega}(z)] < \gamma\) for some constant \(\gamma > 0\) if and only if there exists a symmetric matrix \(X > 0\), such that

\[
\begin{bmatrix}
-X & 0 & XA_0 & XB_0 \\
* & -I & C_0 & 0 \\
* & * & -X & 0 \\
* & * & * & -\gamma^2I
\end{bmatrix} < 0
\]

Denote

\[
G_{0,\omega,\omega}(z) = C_0(zI - A_0)^{-1}B_0
\]

where

\[
A_0 = \begin{bmatrix} A & 0 \\ B_RC_2 & A_F \end{bmatrix}, \quad B_0 = \begin{bmatrix} B_1 \\ B_FD_{21} \end{bmatrix}, \quad C_0 = [C_1 - C_F]
\]

with \(A_F \in \mathbb{R}^{n \times n}\).

Then, we have the following lemma.

**Lemma 3.** Let \(\gamma > 0\) be a given constant. Then, the following statements are equivalent:

1. \(A_{e0}\) is Shur stable, and \([G_{0,\omega,\omega}(z)] < \gamma\);
2. There exists a symmetric positive matrix \(X > 0\), such that

\[
\begin{bmatrix}
-X & 0 & XA_{e0} & XB_{e0} \\
* & -I & C_0 & 0 \\
* & * & -X & 0 \\
* & * & * & -\gamma^2I
\end{bmatrix} < 0
\]

3. There exist a symmetric positive matrix \(X > 0\) and a matrix \(G\), such that

\[
\begin{bmatrix}
X - G - G^T & 0 & G^T\Theta_A & G^T\Theta_B \\
* & -I & C_0 & 0 \\
* & * & -X & 0 \\
* & * & * & -\gamma^2I
\end{bmatrix} < 0
\]

4. There exist matrices \(A_{Fa}, B_{Fa}\), and \(C_{Fa}\), and a symmetric matrix \(P > 0\) with

\[
P = \begin{bmatrix} Y & N \\ N & -N \end{bmatrix}
\]

such that

\[
\begin{bmatrix}
-P & 0 & PA_{Fa} & PB_{Fa} \\
* & -I & C_{Fa} & 0 \\
* & * & -P & 0 \\
* & * & * & -\gamma^2I
\end{bmatrix} < 0
\]

where

\[
A_{Fa} = \begin{bmatrix} A & 0 \\ B_FC_2 & A_{Fa} \end{bmatrix}, \quad B_{Fa} = \begin{bmatrix} B_1 \\ B_FD_{21} \end{bmatrix}
\]

\[
C_{Fa} = [C_1 - C_{Fa}]
\]
5) There exist a symmetric matrix $X > 0$ and a matrix $G$ with structure
\[ G = \begin{bmatrix} Y & N \\ N & -N \end{bmatrix} \] (13)
such that
\[ \begin{bmatrix} X - G - G^T & 0 & G^T A_{eo} & G^T B_{eo} \\ * & -I & C_{eo} & 0 \\ * & * & -X & 0 \\ * & * & * & -\gamma^2 I \end{bmatrix} < 0 \] (14)
holds, where $A_{eo}, B_{eo},$ and $C_{eo}$ are defined by (12).

Theorem 1 and Lemma 3, Theorem 1 also shows that the nonfragile $H_\infty$ filtering problem for system (1).

Proof. By Lemma 3, it is sufficient to show that there exist a matrix $G$ with structure (13) and a symmetric positive matrix $P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix} > 0,$ such that
\[ M_1 = \begin{bmatrix} P - G - G^T & 0 & G^T A_e & G^T B_e \\ * & -I & C_e & 0 \\ * & * & -P & 0 \\ * & * & * & -\gamma^2 I \end{bmatrix} < 0 \] (21)
holds for all $\delta_{aij}, \delta_{bik},$ and $\delta_{clj}$ satisfying (3).

Denote
\[ S = Y + N, \quad \Gamma_1 = \begin{bmatrix} I \\ I \end{bmatrix} \]
\[ \Gamma_1 = \text{diag}(\Gamma_1, I, I_1), I \]
\[ P_{11} = P_{11} + P_{12} + P_{12}^T + P_{22} \]
\[ P_{12} = P_{11} + P_{12}^T, \quad P_{22} = P_{11} \]
Then, (21) is equivalent to
\[ M_2 = \Gamma_1 M_1 \Gamma_1^T = \begin{bmatrix} \Xi_1 & \Xi_2 & 0 & S^T A & S^T A & S^T B_1 \\ \Xi_3 & \Xi_4 & 0 & I & P_{11} & P_{12} \\ * & * & -I & C_1 & 0 \\ * & * & * & -P_{11} & -P_{12} & 0 \\ * & * & * & * & -P_{22} & 0 \\ * & * & * & * & * & -\gamma^2 I \end{bmatrix} < 0 \] (22)
which holds for all $\delta_{aij}, \delta_{bik},$ and $\delta_{clj}$ satisfying (3), where $\Xi_1, \Xi_2, \Xi_3,$ and $\Xi_4$ are defined by (18), and
\[ \Pi_1 = (S - N)^T A + N^T (B_F C_2 + \Delta B_F C_2 + A_F + \Delta A_F) \]
\[ \Pi_2 = (S - N)^T A + N^T (B_F C_2 + \Delta B_F C_2) \]
\[ \Pi_3 = (S - N)^T B_1 + N^T (B_F D_2 + \Delta B_F D_2) \]
Obviously, $M_2$ is convex for each $\delta_i, \delta_i \in \{\delta_{aij}, \delta_{bik}, \delta_{clj}\}$ satisfying (3)), so (22) is equivalent to
\[ M_3 = \begin{bmatrix} \Xi_1 & \Xi_2 & 0 & S^T A & S^T A & S^T B_1 \\ \Xi_3 & \Xi_4 & 0 & I & P_{11} & P_{12} \\ * & * & -I & C_1 & 0 \\ * & * & * & -P_{11} & -P_{12} & 0 \\ * & * & * & * & -P_{22} & 0 \\ * & * & * & * & * & -\gamma^2 I \end{bmatrix} < 0 \] (23)
for all $\delta_{aij}, \delta_{bik},$ and $\delta_{clj} \in \{-\delta_a, \delta_b\}, \ i, j = 1, \ldots, n; \ k = 1, \ldots, p; \ l = 1, \ldots, q$. By (20), (21) and the Schur complement, it is concluded that (23) is equivalent to (19).

Remark 2. Theorem 1 and Lemma 3, Theorem 1 also shows that the nonfragile $H_\infty$ filtering problem becomes a convex one when the state-space realizations of the designed filters with gain variations admit the slack variable matrix $G$ with the structure of (13). For the case that the designed filter contains
\[ A_F = (N^T)^{-1} F_A, \quad B_F = (N^T)^{-1} F_B, \quad C_F = C_F \] (20)
no gain variations, from Lemma 3, it follows that the design condition given in Theorem 1 reduces a necessary and sufficient condition for the standard $H_{\infty}$ filtering problem, which means that the structure constraint (13) on the slack matrix $G$ does not result in any conservativeness for the standard $H_{\infty}$ filter design.

For the nonfragile filter design method given by Theorem 1, it should be noted that the number of LMIs involved in (19) is $2^{n+np+nq}$, which results in the difficulty of implementing the LMI constraints in computation. For example, when $n = 5$ and $p = q = 1$, the number of LMIs involved in (19) is $2^{35}$, which already exceeds the capacity of the current LMI solver in Matlab. To overcome the difficulty raising from implementing the design condition given in Theorem 1, the following method is developed.

Denote

\[
F_{a1} = [f_{a11} f_{a12} \cdots f_{a1na}],
\]

\[
F_{a2} = [f_{a21} f_{a22} \cdots f_{a2na}]^T
\]

where $l_a = n^2 + np + nq$, and

\[
f_{abk1} = \left[01_{nx} \ (N_{T}e_i) \ 0_{1xq} \ 0_{1nx} \ 0_{1xq} \right]^T
\]

\[
f_{abk2} = \left[01_{nx} \ 0_{1nx} \ 0_{1xq} \ e_j^T \ 0_{1nx} \ 0_{1xq} \right]^T
\]

for $k = (i-1)n + j$, $i, j = 1, \ldots, n$.

\[
f_{abk1} = \left[01_{nx} \ (N_{T}e_i) \ 0_{1xq} \ 0_{1nx} \ 0_{1xq} \right]^T
\]

\[
f_{abk2} = \left[01_{nx} \ 0_{1nx} \ 0_{1xq} \ e_j^T \ 0_{1nx} \ 0_{1xq} \right]^T
\]

for $k = n^2 + (i-1)p + j$, $i = 1, \ldots, p$, $j = 1, \ldots, n$.

\[
f_{abk1} = \left[01_{nx} \ 0_{1nx} \ -g_i^T \ 0_{1nx} \ 0_{1xq} \right]^T
\]

\[
f_{abk2} = \left[01_{nx} \ 0_{1nx} \ 0_{1xq} \ e_j^T \ 0_{1nx} \ 0_{1xq} \right]^T
\]

for $k = n^2 + np + (i-1)n + j$, $i = 1, \ldots, p$, $j = 1, \ldots, n$.

Let $k_0, k_1, \ldots, k_{sa}$ be integers satisfying

\[k_0 = 0 < k_1 < \cdots < k_{sa} = l_a\]

and matrix $\Theta$ have the following structure

\[
\Theta = \begin{bmatrix}
\text{diag}(\theta_{11}, \ldots, \theta_{1n}) & \text{diag}(\theta_{12}, \ldots, \theta_{12}) \\
\text{diag}(\theta_{12}, \ldots, \theta_{2n}) & \text{diag}(\theta_{22}, \ldots, \theta_{22})
\end{bmatrix}
\]

where $\theta_{11}, \theta_{12}, \ldots, \theta_{12}$ are matrices in $\mathbb{R}^{(k_i - k_{i-1}) \times (k_i - k_{i-1})}$, $i = 1, \ldots, sa$. Then, we have Theorem 2.

**Theorem 2.** Considering system (1), let $\gamma > 0$ and $\delta_a > 0$ be given constants. If there exist matrices $F_a, F_B, C_f, S, N, P_{21}, P_{11} > 0, P_{22} > 0$ and symmetric matrix $\Theta$ with the structure described by (25), such that the following LMIs hold

\[
\begin{bmatrix}
Q & F_{a1} \\
F_{a1}^T & F_{a2}
\end{bmatrix} < 0
\]

\[
I - \begin{bmatrix}
\text{diag}(\delta_{k_{i-1}+j}, \ldots, \delta_{k_i}) \\
\text{diag}(\delta_{k_{i-1}+j}, \ldots, \delta_{k_i})^T
\end{bmatrix}
\begin{bmatrix}
\theta_{11} \\
\theta_{22}
\end{bmatrix}
\begin{bmatrix}
\theta_{11}^T \\
\theta_{22}
\end{bmatrix}
\geq 0, \quad j = 1, \ldots, k_i - k_{i-1}, \quad i = 1, \ldots, sa
\]

\[
\Theta
\]

\[
\begin{bmatrix}
\Xi_1 & \Xi_2 & 0 & S^TA \\
* & \Xi_3 & 0 & (S - N)^TA + F_B C_2 + F_A \\
* & * & -I & C_1 - C_f \\
* & * & * & -P_{11}
\end{bmatrix}
\]

\[
Q = \begin{bmatrix}
S^TA & S^TB_1 \\
(S - N)^TA + F_B C_2 & (S - N)^TB_1 + F_B D_{21}
\end{bmatrix}
\]

\[
C_1 \\
-P_{11} \\
P_{22} \\
-\gamma^2 I
\]

with $\Xi_1, \Xi_2, \Xi_3$ defined by (18). Then, filter (2) with additive uncertainty described by (3) and the filter gain parameters given by (20) solves the nonfragile $H_{\infty}$ filtering problem for system (1).

**Proof.** Obviously, (19) can be written as

\[
M_0 = Q + \Delta Q + \Delta Q^T < 0
\]

where

\[
\Delta Q = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \Delta Q_1 & \Delta Q_2 & \Delta Q_3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

with

\[
\Delta Q_1 = \sum_{i,j=1}^{n} \delta_{aij} N^T e_i e_i^T + \sum_{i=1}^{p} \sum_{k=1}^{n} \delta_{bk} N^T e_i h_k^T C_2
\]

\[
\Delta Q_2 = \sum_{i=1}^{n} \sum_{k=1}^{p} \delta_{bk} N^T e_i h_k^T C_2
\]

\[
\Delta Q_3 = \sum_{i=1}^{n} \sum_{k=1}^{p} \delta_{bk} N^T e_i h_k^T D_{21}, \quad \Delta Q_4 = -\sum_{i=1}^{q} \sum_{j=1}^{p} \delta_{cj} \theta_{1j}^T e_j^T
\]

for all $\delta_{aij}, \delta_{bk}, \delta_{cj} \in \{-\delta_a, \delta_a\}$, $i = 1, \ldots, p$; $k = 1, \ldots, p$; and $l = 1, \ldots, q$.

By using (24), it follows that (29) is equivalent to

\[
M_0 = Q + \sum_{i=1}^{l_a} \delta_{a1i} f_{a2i} + \sum_{i=1}^{l_a} \delta_{a2i} f_{a2i}^T = Q + F_{a1} \Delta a_{sa} F_{a2} + (F_{a1} \Delta a_{sa} F_{a2})^T < 0
\]

holds, where $\Delta a_{sa} = \text{diag}(\delta_{ai}, \ldots, \delta_{ai})$ for all $\delta_i \in \{-\delta_a, \delta_a\}$. By Lemma 4, it follows that (30) is further equivalent to a symmetric matrix $\Theta \in \mathbb{R}^{l_a \times l_a}$, such that (26) and

\[
I \geq 0
\]

hold for all $\delta_i \in \{-\delta_a, \delta_a\}, i = 1, \ldots, l_a$. Notice that the set of $\Theta$ satisfying (25) is a subset of the set of $\Theta$ satisfying (31), hence, the conclusion follows.

**Remark 3.** From the proof of Theorem 2, it follows that when $s_a = 1$, the set of $\Theta$ satisfying (25) is equal to the set of $\Theta$ satisfying (31), and the design conditions
given in Theorem 2 and Theorem 1 are equivalent. \( \Theta \) satisfying (26) and (31) (or (27) with \( s_a = 1 \)) is said to be a vertex separator.[11] Notice that the number of LMIs involved in (31) or (27) with \( s_a = 1 \) is still \( 2^{n^2+n+p+1} \), so that the difficulty of implementing the LMI constraints remains.

To overcome the difficulty, Theorem 2 presents a sufficient condition for the nonfragile \( H_\infty \) filter design in terms of separator \( \Theta \) with the structure described by (25), where the number of LMIs involved in (27) is \( \sum_{i=1}^{n^2} 2^{k_i-1} \), which can be controlled not to grow exponentially by reducing the value of max \( (k_i - k_{i-1}) \), \( i = 1, \ldots, s_a \). Compared with the \( \Theta \) being of full block in (26) and (27), \( \Theta \) with the structure described by (25) satisfying (26) and (27) is said to be a structured vertex separator. However, it should be noted that the design condition given in Theorem 2 may be more conservative than that given in Theorem 1 because of the structure constraint on \( \Theta \). But the design condition proposed in Theorem 2 solves the numerical computation problem, which cannot be solved by the design condition given in Theorem 1. On the other hand, in Theorem 2, the smaller the value of \( s_a \) is, the less conservativeness is introduced.

Remark 4. Obviously, the conditions for designing nonfragile \( H_\infty \) filters given in Theorem 1 and Theorem 2 can be easily extended to deal with the robust nonfragile \( H_\infty \) filtering problem for systems with polytopic uncertainties too, because the system matrices are affinely involved in the proposed design conditions.

2.2 Comparison with the existing design method

In the following, we will introduce the result of nonfragile \( H_\infty \) filter design with norm-bounded gain variations. And at the same time, the relationship with our result is illustrated.

Similar to [5] and [6] for nonfragile filter design with norm-bounded uncertainty, the norm-bounded type of gain variations \( \Delta A_F, \Delta B_F, \) and \( \Delta C_F \) can be overbounded[12] by the following norm-bounded uncertainty

\[
\Delta A_F = M_a F_1(t) E_a, \quad \Delta B_F = M_b F_2(t) E_b, \quad \Delta C_F = M_c F_3(t) E_c
\]

where

\[
M_a = [M_{a1} \cdots M_{an}], \quad E_a = [E_{a1}^T \cdots E_{an}^T]^T
\]

\[
M_b = [M_{b1} \cdots M_{bn}], \quad E_b = [E_{b1}^T \cdots E_{bn}^T]^T
\]

\[
M_c = [M_{c1} \cdots M_{cn}], \quad E_c = [E_{c1}^T \cdots E_{cn}^T]^T
\]

with

\[
M_{ak} = e_i, \quad E_{ak} = e_j^T
\]

for \( k = (i-1)n + j, \ i, j = 1, \ldots, n \)

\[
M_{bk} = g_i, \quad E_{bk} = h_j^T
\]

for \( k = n^2 + (i-1)p + j, \ i = 1, \ldots, p, \ j = 1, \ldots, n \)

\[
M_{ck} = f_i, \quad E_{ck} = e_j^T
\]

for \( k = n^2 + np + (i-1)n + j, \ i = 1, \ldots, q, \ j = 1, \ldots, n \)

and \( F_i^T(t) F_i(t) \leq \delta_i^2 I, \) for \( i = 1, 2, 3, \) represent the uncertain parameters, here \( \delta_i \) is the same as before.

To facilitate the presentation, we denote \( \hat{F}_A = \hat{N} A_F, \quad \hat{F}_B = \hat{N} B_F \)

\[
M_{a1} = \begin{bmatrix}
N M_a & N M_b \\
0 & -M_c
\end{bmatrix}
\]

\[
M_{a2} = \begin{bmatrix}
0 & 0 & 0 & E_a & 0 & 0 \\
0 & 0 & 0 & 0 & E_c C_2 & E_c C_2 E_c D_{21}
\end{bmatrix}
\]

By using the method in [5 - 6], the nonfragile \( H_\infty \) filter design with norm-bounded gain variations is reduced to solving the following LMI

\[
\begin{bmatrix}
Q & M_{a1} & \delta_a \varepsilon M_{a2}^T \\
* & -\varepsilon I & 0 \\
* & * & -\varepsilon I
\end{bmatrix} < 0
\]

(33)

with matrix variables \( \hat{S} > 0, \hat{N} < 0, \hat{e} > 0, \) where

\[
Q = \begin{bmatrix}
-\hat{S} & \hat{S} & 0 & \hat{S} & 0 & \hat{S} \\
-\hat{S} & \hat{S} & 0 & \hat{S} & 0 & \hat{S} \\
0 & 0 & Q_1 & Q_2 & Q_3 \\
0 & 0 & 0 & \hat{N} & 0 \\
0 & 0 & 0 & 0 & -\varepsilon^2 I
\end{bmatrix}
\]

with \( Q_1 = (\hat{S} - \hat{N}) A + F_1 + F_2 C_2, \ Q_2 = (\hat{S} - \hat{N}) A + F_2 C_2, \) and \( Q_3 = (\hat{S} - \hat{N}) B_1 + F_1 D_{21} + F_2 D_{22}. \)

The following Lemma will show the relationship between condition (33) and the condition for designing nonfragile \( H_\infty \) filters given in Theorem 2.

Lemma 5. Considering system (1), if condition (33) is feasible, then, the condition for designing nonfragile \( H_\infty \) filters given in Theorem 2 is feasible.

Proof. Let \( S = F_1 = \hat{S}, \ N = \hat{N}, \) and \( F_2 = \hat{S} - \hat{N} > 0. \) Then, it is easy to see that \( Q = \hat{Q}, \ F_{a1} = M_{a1}, \) and \( F_{a2} = M_{a2}, \) i.e., condition (33) becomes

\[
\begin{bmatrix}
Q & F_{a1} & \delta_a \varepsilon F_{a2}^T \\
* & -\varepsilon I & 0 \\
* & * & -\varepsilon I
\end{bmatrix} < 0
\]

(34)

In Theorem 2, when \( s_a = l_a, \) according to (34) and \( F_{it}^T(t) F_{it}(t) \leq \delta_i^2 I, \ i = 1, 2, 3, \) and by the Schur complement, there exists a matrix \( \Theta \) with the structure \( \Theta = \begin{bmatrix} \varepsilon \delta_a^2 I & 0 \end{bmatrix} \) such that the following LMIs hold

\[
\begin{bmatrix}
Q & F_{a1} & 0 \\
F_{a1}^T & 0 & I \\
0 & I
\end{bmatrix} \Theta \begin{bmatrix} F_{a2} & 0 \end{bmatrix} = \begin{bmatrix} Q + \varepsilon \delta_a^2 F_{a2} & F_{a1} & 0 \\
0 & -\varepsilon I & 0 \\
F_{a2}^T & 0 & I
\end{bmatrix} < 0
\]

(35)

\[
\begin{bmatrix}
I & \theta_{11} & \theta_{12} \\
\theta_{11}^T & \theta_{22} \\
\theta_{12}^T & \delta_1
\end{bmatrix} < \varepsilon \delta_a^2 - \varepsilon \delta_i^2 \geq 0
\]

for all \( i = 1, \ldots, l_a \)

Remark 5. From the proof of Lemma 5, it follows that condition (33) is more conservative than the nonfragile \( H_\infty \) filter existence condition given in Theorem 2 with \( s_a = l_a. \) However, as indicated in Remark 3, the case of \( s_a = l_a \) is the worst case of the method. So the existing nonfragile \( H_\infty \)
filter design method with the norm-bounded gain variations is more conservative than the nonfragile $H_\infty$ filter design method proposed in this paper.

2.3 Evaluation of $H_\infty$ performance index

In Theorem 2, we restrict the slack variable matrix $G$ with the structure (13) for obtaining the convex design condition, which may result in more conservative evaluation of the $H_\infty$ performance index bound. So, in this subsection, for a designed filter, the matrix $G$ without the restriction is exploited for obtaining less conservative evaluation of the $H_\infty$ performance index bound.

When the filter parameter matrices $A_F, B_F,$ and $C_F$ are known, the problem of minimizing $\gamma$ subject to (3) for a given $\delta_s > 0$ and $\| G_{z,\omega}(z) \| < \gamma$ can be converted into the one, where $\gamma^2$ is minimized subject to the following LMIs

$\begin{bmatrix}
P - G - G^T & 0 & G^T A_F & G^T B_0 \\
* & -I & C_F & 0 \\
* & * & -P & 0 \\
* & * & * & -\gamma^2 I
\end{bmatrix} < 0 \quad (35)$

for all $\delta_{aij}, \delta_{bk}, \delta_{ct} \in \{-\delta_s, \delta_s\}$,

$i, j = 1, \ldots, n; \ k = 1, \ldots, p; \ l = 1, \ldots, q$

where $A_F, B_0,$ and $C_F$ are defined as in (5).

Similar to the design condition given in Theorem 1, the above method is also with the numerical computation problem. To solve the problem, the following lemma provides a solution using the structured vertex separator approach.

Denote

$G_{a_1} = [g_{a_11} \quad g_{a_12} \cdots \quad g_{a_{11n}}]^T$

$G_{a_2} = [g_{a_21} \quad g_{a_22} \cdots \quad g_{a_{21n}}]^T \quad (36)$

where

$g_{a_{1k}} = [(0_{1 \times n} \quad e_1^T) G \quad 0_{1 \times q} \quad 0_{1 \times 2n} \quad 0_{1 \times r}]^T$

$g_{a_{2k}} = [0_{1 \times 2n} \quad 0_{1 \times q} \quad 0_{1 \times n} \quad e_1^T \quad 0_{1 \times r}]^T$

for $k = (i - 1)n + j, \ i, j = 1, \ldots, n$

$g_{a_{1k}} = [(0_{1 \times n} \quad e_1^T) G \quad 0_{1 \times q} \quad 0_{1 \times 2n} \quad 0_{1 \times r}]^T$

$g_{a_{2k}} = [0_{1 \times 2n} \quad 0_{1 \times q} \quad h_1^T C_2 \quad 0_{1 \times n} \quad h_1^T D_{21}]^T$

for $k = n^2 + (i - 1)p + j, \ i = 1, \ldots, n; \ j = 1, \ldots, p$

$g_{a_{1k}} = [0_{1 \times n} \quad -g_1^T \quad 0_{1 \times 2n} \quad 0_{1 \times r}]^T$

$g_{a_{2k}} = [0_{1 \times 2n} \quad 0_{1 \times q} \quad 0_{1 \times n} \quad e_1^T \quad 0_{1 \times r}]^T$

for $k = n^2 + np + (i - 1)n + j, \ i = 1, \ldots, n; \ j = 1, \ldots, n$

Then, we have Lemma 6.

Lemma 6. Considering system (1), let $\gamma > 0$ and $\delta_s > 0$ be constants and filter parameter matrices $A_F, B_F,$ and $C_F$ be given. Then, $\| G_{z,\omega}(z) \| < \gamma$ holds for all $\delta_{aij}, \delta_{bk}$ and $\delta_{ct}$ satisfying (3), if there exist a matrix $G,$ a positive definite matrix $P > 0,$ and a symmetric matrix $\Theta$ with the structure described by (25), such that (27) and the following LMI hold

$\begin{bmatrix}
Q_s & G_{a_1} \\
G_{a_1}^T & 0
\end{bmatrix} + \begin{bmatrix}
G_{a_2} & 0 \\
0 & I
\end{bmatrix}^T \Theta \begin{bmatrix}
G_{a_2} & 0 \\
0 & I
\end{bmatrix} < 0 \quad (37)$

where

$Q_s = \begin{bmatrix}
P - G - G^T & 0 & G^T A_F & G^T B_0 \\
* & -I & C_F & 0 \\
* & * & -P & 0 \\
* & * & * & -\gamma^2 I
\end{bmatrix}$

with $A_0, B_0,$ and $C_0$ defined by (7).

Proof. By using (35) and (36), the proof is obtained similar to Theorem 2, and therefore, it is omitted here. \[\square\]

Remark 6. For evaluating the $H_\infty$ performance bound of the transfer function from $\omega$ to $z_\omega,$ the condition given in Lemma 6 is usually less conservative than that given in Theorem 2 because no structure constraint on the slack variable matrix $G$ in Lemma 6 is imposed.

3 Example

To illustrate the effectiveness of the designed nonfragile $H_\infty$ filter, an example is given to provide a comparison between the proposed nonfragile $H_\infty$ filter design method and the existing nonfragile $H_\infty$ filter design method with the norm-bounded gain variations. We consider a linear system of the form (1) with

$A = \begin{bmatrix}
0 & 1 & -0.5 \\
-1 & -0.5 & 1 \\
-1 & 0 & 1
\end{bmatrix}, \quad B_1 = \begin{bmatrix}
-1 & 0 \\
0.5 & 0 \\
-1 & 0
\end{bmatrix}$

$C_1 = [1 -1 1], \quad C_2 = [-1 \quad 0.5 \quad 2], \quad D_{21} = [0 \quad 0.9]$.

In the case where the designed filter contains no gain variations, by the standard $H_\infty$ filtering method for discrete-time systems, the optimal $H_\infty$ performance index of the standard closed-loop system is achieved as $\gamma_{\text{opt}} = 3.7282$.

3.1 Existing method given by condition (33)

In this subsection, we design a nonfragile $H_\infty$ filter by using condition (33).

Assume that the designed filter is with norm-bounded additive uncertainties described by (32), by applying condition (33) with $\delta_s = 0.05,$ a nonfragile $H_\infty$ filter $F_{\text{norm}}$ is designed and the $H_\infty$ performance index of the obtained nonfragile $H_\infty$ filter is $4.7319$.

3.2 New method given by Theorem 2

In the following, we design a nonfragile $H_\infty$ filter by using Theorem 2.

Assume that the designed filter is with the additive uncertainties described by (3). For this case, it is difficult to apply Theorem 1 to design a nonfragile $H_\infty$ filter because the number of the LMI constraints involved in (19) is $3^{15}$. However, Theorem 2 is applicable to this case. By applying Theorem 2 with $\delta_s = 0.05$ and $k_i = i, i = 1, \ldots, 15,$ i.e., $s_a = 15$ as well as $k_1 = 3i, i = 1, \ldots, 5,$ i.e., $a_s = 5,$ the nonfragile filters $F_{in15}$ and $F_{in5}$ are designed, and the $H_\infty$ performance indices of the obtained filters are $4.1791 (s_a = 15)$ and $4.1790 (s_a = 5)$, respectively.

3.3 Comparison

In this subsection, tables are given to provide comparison results between the nonfragile $H_\infty$ filters designed by the proposed method (Theorem 2) and the nonfragile $H_\infty$ filter designed by the existing method (condition (33)).

First, Table 1 offers an $H_\infty$ performance comparison between the two design methods given by condition (33) and Theorem 2 with $\delta_s = 0.05$.

<table>
<thead>
<tr>
<th>Condition (33)</th>
<th>Theorem 2 ($s_a = 15$)</th>
<th>Theorem 2 ($s_a = 5$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\gamma$</td>
<td>4.7319</td>
<td>4.1791</td>
</tr>
</tbody>
</table>

From Table 1, it is easy to see that compared with the optimal $H_\infty$ performance index bound $\gamma_{\text{opt}} = 3.7282$, the performance indices of the filter designed by the existing
method (condition (33)) is degraded by 26.92%. The performance indices of the filters designed by the proposed method (Theorem 2) are degraded as the same as 12.99% (for $s_a = 15$ or $s_a = 5$), which are much more improved than 26.92%.

For the designed filters, Lemma 6 gives better performance indices shown in Table 2.

Table 2 Performance index evaluation by Lemma 6 with $\delta_a = 0.95$

<table>
<thead>
<tr>
<th>$\gamma$ ($s_a = 15$)</th>
<th>$F_{\infty,\text{norm}}$</th>
<th>$F_{\infty,15}$</th>
<th>$F_{\infty,5}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\gamma$ ($s_a = 5$)</td>
<td>4.3539</td>
<td>4.1069</td>
<td>--</td>
</tr>
<tr>
<td></td>
<td>4.3439</td>
<td>--</td>
<td>4.1060</td>
</tr>
</tbody>
</table>

Obviously, compared with the optimal $H_\infty$ performance indices $\gamma_{\text{opt}} = 4.7282$, by using Lemma 6, the $H_\infty$ performance indices of the nonfragile $H_\infty$ filter $F_{\infty,\text{norm}}$ are degraded by 16.78% for $s_a = 15$ and 16.51% for $s_a = 5$. Correspondingly, the performance indices for the nonfragile $H_\infty$ filters $F_{\infty,15}$ and $F_{\infty,5}$ are degraded by 10.16% for $s_a = 15$ and 10.13% for $s_a = 5$, respectively. These numerical results show the superiority of our nonfragile filtering method.

4 Conclusions

In this paper, the problem of nonfragile $H_\infty$ filter design for linear discrete-time systems has been addressed, where the filter to be designed is assumed to be with additive gain variations of interval type due to FWL effects. A notion of structured vertex separator is proposed to deal with the problem and exploited to develop sufficient conditions for the nonfragile $H_\infty$ filter design in terms of solutions to a set of LMIs. Moreover, to reduce the design conservativeness, the slack variable method is adopted to realize the decoupling between the Lyapunov matrix and the system dynamic matrix. A comparison between our method and the existing method for nonfragile $H_\infty$ filter design is presented to indicate the superiority of our proposed method. The numerical example has shown the effectiveness of the proposed approach.

References

1 de Oliveira M C, Geromel J C. $H_2$ and $H_\infty$ filtering design subject to implementation uncertainty. SIAM Journal on Control and Optimization, 2006, 44(2): 515–539


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