Measurement-feedback Control for Systems with Input and Measurement Delays

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Abstract The paper concentrates on finding the $H_{\infty}$ measurement-feedback control-law for systems with not only an input-delay but also a measurement-delay. Krein space, together with pseudo-measurements, is introduced so that the reorganizing technique can be utilized after we convert the original problem into a minimizing one. Finally, we get the desired controller and see that the separation principle is also applicable to delay systems to some extent.

Key words Krein space, $H_{\infty}$ control, measurement-feedback, reorganizing innovation

Because of the presence of uncertain exogenous disturbances and model uncertainties, the $H_{\infty}$ control has intrigued many researchers for decades. The study can be traced back to 1981, when it was originally proposed by Zames[1]. Research related to the $H_{\infty}$ control stepped into the new epoch as the state-space idea was introduced by Doyle[2]. There are abundant results related to the $H_{\infty}$ control in time domain[3,4,5] and frequency domain[6,7,8,9]. As a late comer, the time-domain approach gave an impetus to research the $H_{\infty}$ control problems for time-varying systems, nonlinear systems[10,11], delay systems[12,13,14,15], stochastic systems and so on.

However, it seems hard to find results related to the $H_{\infty}$ measurement-feedback control-law for systems with I/O delays except a few works[16,17,18]. Mirkin[19] and Meinsma[20] first investigated systems with single I/O delay. Several years later, they generalized nontrivially the single-delay study to the multiple I/O delay case[21]. Transfer functions are the key medium[22,23] to solve the $H_{\infty}$ control problem. Unfortunately, they are “aliens” for the time-varying systems. Liu’s idea[24] is applicable to the time-varying systems, but it is only allowed for systems with the delay-measurement.

The paper considers the systems with measurement-delay and input-delay. It needs no transfer function and delay-measurement.

The $H_{\infty}$ control problem under investigation is stated as: given a scalar $\gamma > 0$ and observations $\langle y(t) \rangle_{t=0}^{\infty}$, find a finite-horizon measurement-feedback control strategy for

\[ y_i(t) = H_1 x(t_d) + v_i(t) \]  

where $t_d = t - d$. In the following, similar denotation will take the same meaning in the rest paper. $x(t) \in \mathbb{R}^n$, $w(t) \in \mathbb{R}^m$, $u_i(t) \in \mathbb{R}_+$, $v_i(t), y_i(t), y(t) \in \mathbb{R}^r$, and $z(t) \in \mathbb{R}^r$ represent the state, input noise, control inputs, measurement noise, measurement outputs, and the signal to be regulated, respectively.

For convenience, denote

\[ \Xi_j = \tau_d - j, \Theta_j = \tau_d + j, \Xi_i = \tau_d - i, \Theta_i = \tau_d + i \]  

\[ \mathbb{N}_j = N - j, \mathbb{N}_i = N - i, \mathbb{N}_d = N_d - j, \mathbb{N}_d = N_d - i \]

\[ K_j^{41} = \begin{bmatrix} K^j_0 & K^j_1 \\ H_0 & H_1 \end{bmatrix}, K_j^{42} = \begin{bmatrix} K^j_0 & 0 \\ H_0 & 0 \end{bmatrix}, K_j^{21} = \begin{bmatrix} K^j_0 \\ H_0 \end{bmatrix} \]

\[ K_j^{31} = \begin{bmatrix} K^j_0 & 0 \\ H_0 & H_1 \end{bmatrix}, \mathcal{M}_j^{11} = \begin{bmatrix} K^j_0 \\ H_0 \end{bmatrix}, \mathcal{M}_j^{32} = \begin{bmatrix} K^j_0 & 0 \\ H_0 & 0 \end{bmatrix} \]

1 Problem statement

Consider a linear system described by the following discrete-time model

\[ x(t + 1) = \Phi x(t) + B_0 u_0(t) + B_1 u_i(t_d) + G w(t) \]

\[ y_0(t) = H x(t) + v_0(t) \]

Notations. Throughout the paper, $\langle x, y \rangle$ denotes the inner product of vector $x, y$; $\text{col}[x_1, x_2, \ldots, x_n]$ denotes the column vector formed by stacking all the vectors $x_1, x_2, \ldots, x_n$; and 0 denotes the zero matrix of $s \times s$.

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2 Construction of ideal quadratic form

In this section, we shall make a crucial preparation for solving the $H_\infty$ measurement-feedback control problem on the basis of the results\cite{16}. For convenience, we first introduce a couple of Riccati equations\cite{16} as

$$P_j = \Phi^T P_{j+1} \Phi + Q - \Phi^T P_{j+1} \Gamma_j (R_j + \Gamma_j^T P_{j+1} \Gamma_j)^{-1} \Gamma_j^T P_{j+1} \Phi,
$$

\[ j = N, N-1, \ldots, 0, \ P_{N+1} = 0 \]  

(11)

$$P_j' = \Phi^T P_{j+1} \Phi + Q - \Phi^T P_{j+1} \Gamma_j (R_j + \Gamma_j^T P_{j+1} \Gamma_j)^{-1} \Gamma_j^T P_{j+1} \Phi,
$$

\[ j = \min\{d, N+1\}, \ldots, 0, \ P_{N+1}' = 0, \text{ or } P_0' = P_{d+1} \]  

(12)

Obviously, $P_j' = P_{j+d}$ as $j \geq d$, where

$$Q = C^T C, \quad \Gamma_j = \begin{bmatrix} [B_0, G], & 0 \leq j < d \\ [B_0, G; B_1], & j \geq d \end{bmatrix}
$$

\[ R_j = \begin{cases} \text{diag}(D_{0}^T D_0, -\gamma^2 I_1), & 0 \leq j < d \\ \text{diag}(D_{0}^T D_0, -\gamma^2 I_1, D_1^T D_1), & j \geq d \end{cases} \]

Assume that the above Riccati equations have bounded solutions, and denote

$$M_j' = R_j + \Gamma_j^T P_{j+1} \Gamma_j (M_j')^{-1}, \quad K_j' = \Phi^T P_{j+1} \Gamma_j (M_j')^{-1}$$

(13)

throughout the paper.

**Remark 1.** It should be noted that

1) Despite the same structures of the above two Riccati equations, they still have different solutions because of different initial values and the range of $j$.

2) The two equations have the same structures for linear time invariant (LTI) systems. Nevertheless, for linear time varying (LTV) systems, they just have similar structure in the one by not 16 but the same one.

Considering the performance index (9), we define

$$J_N^\infty = z^T(0) \Pi_0^T z(0) + \| z \|_{[0, N]}^2 - \gamma^2 J_N
$$

(14)

where

$$J_N = \| z \|_{[0, N]}^2 - \gamma^2 \| w \|_{[0, N]}^2
$$

(15)

It is clear that the $H_\infty$ controller $u_i(t)$ satisfies (9) if and

only if it renders that $J_N^\infty$ in (14) is positive for all non-zero \( z(0); u(t), u_i(t), 0 \leq t \leq N, v_i(t), d \leq t \leq N \).

It is not hard to see that (14) is almost the same as (9) in [16], but the additional term \( \| w \|_{[0, N]}^2 \) is only involved by the measurement equations; what is more important is that (6) in [16] and (15) are totally identical. Therefore, in view of Lemma 7 in [16], (14) is rewritten as

$$J_N^\infty = z^T(0) \Pi_0^T z(0) + \| u \|_{[0, N]}^2 - 
\gamma^2 \sum_{\tau=0}^{N} [u_i(\tau) - \hat{u}_i(\tau)]^T K \hat{u}_i [u_i(\tau) - \hat{u}_i(\tau)] 
$$

(16)

where

$$v_i(\tau) = \text{col}\{u(\tau), w(\tau)\}
$$

(17)

with

$$u(t) = \begin{cases} \text{col}\{u_0(\tau), u_i(\tau)\}, & 0 \leq \tau \leq N_d \\ \text{col}\{u(\tau)\}, & \tau > N_d \end{cases}
$$

(18)

$$v_i(\tau) \text{ is obtained from } u_i(\tau) \text{ with } u_i(\tau) \text{ and } u_i(\tau)(i=0, 1) \text{ replaced by } u(\tau) = [0_m, P_0 \bar{b}_0(\tau)], u_i(\tau) = [I_m, 0_0 \bar{b}_0(\tau)], \text{ and } u_i(\tau) = \bar{v}_i(\tau), \text{ respectively, while } \bar{v}_i(\tau) \text{ is given by}
$$

$$\bar{v}_0(\tau) = -[P_0\bar{b}_0(\tau)]^T \bar{x}(\tau) - 
\sum_{k=1}^{d} [F_k(0)]^T B_1 \bar{u}_1(k-1 + \tau_d)
$$

(19)

$$\bar{v}_1(\tau) = -[0_m + p \bar{m}_n \bar{b}_0(\tau)] \left\{ \begin{array}{l} F_k^T (\tau) \bar{x}(\tau) + \\ \sum_{k=1}^{d} [F_k \bar{b}(d)]^T B_1 \bar{u}_1(k-1 + \tau_d) \end{array} \right\}
$$

(20)

In (19) and (20),

$$F_k^T (t) = \left\{ \begin{array}{l} P_{k}^T (\Phi_{i+1,k}^T \Gamma_1 (M_i')^{-1} - (\Phi_{i,k}^T)^T G^* (k) K_i), \\ (I_n - P_{k}^T G^* (k)) \Phi_{j,k} K_i', \quad k \leq t \leq N \end{array} \right\}
$$

(21)

with

$$\Phi_i^T = \Phi^T - K_i^T \Gamma_i, \quad \Phi_i' = I, \quad \Phi_{j,k} = \Phi_j \cdots \Phi_{j-1}, \quad i \geq j \geq 0
$$

(22)

$$G^* (k) = \sum_{j=1}^{k} (\Phi_{j,k}^T \Gamma_{j-1} (M_{j-1}')^{-1} \Gamma_{j-1} \Phi_{j,k}^T), \quad G^* (0) = 0
$$

(23)

and $K_i'$ is defined as in (13). A careful observation will show us that $\Phi_{j,k}'$ is actually the state transition matrix corresponding to $\Phi_{j,k}'$.

Next, we will provide an important parameter $\bar{M}_v$, in \( (16) \), For $0 \leq \tau \leq N_d,$

$$\bar{M}_v = \Theta_v^T \bar{P}_{v+1} \Theta_v + \text{diag} \{ D_0^T D_0; d_1^T d_2; \gamma^2 I_p \}
$$

(24)

For $N_d < \tau \leq N,$

$$\bar{M}_v = \Theta_v^T \bar{P}_{v+1} \Theta_v + \text{diag} \{ D_0^T D_0; \gamma^2 I_p \}
$$

(25)

In the above, $\Theta_v = \begin{bmatrix} \text{diag} \{ B_0, B_1; G \}, & 0 \leq \tau \leq N_d \\ \text{diag} \{ B_0, G \}, & N_d < \tau \leq N \end{bmatrix}
$$

(26)

and $\bar{P}_v(i,j)(i \geq j)$ is given by

$$\bar{P}_v(0,0) = P_0^T, \quad \bar{P}_v(1,0) = P_0^T (\Phi_{0,0}^T)^T, \quad \bar{P}_v(1,1) = P_0^T (I - G^* (d) \Phi_{0,1}^T)
$$

(27)

where $\Phi_{0,d}^T$ and $G^* (d)$ are defined as in (22) and (23), respectively, and $P_0^T$ and $P_0^T$ can be obtained from (12).
Let

\[
R_w = \begin{bmatrix}
I_{r_I}, \\
\text{diag}(I_{r_I}, I_{r_I})
\end{bmatrix},
\quad 0 \leq \tau < d
\]
\[H^* = \begin{bmatrix}
H_0, \\
\text{diag}(H_0, H_1)
\end{bmatrix},
\quad 0 \leq \tau < d
\]
\[
x(\tau) = \begin{bmatrix}
x(\tau), \\
\text{col}(x(\tau), x(\tau_0))
\end{bmatrix},
\quad 0 \leq \tau < d
\]
\[
y(\tau) = \begin{bmatrix}
y_0(\tau), \\
\text{col}(y_0(\tau), y_1(\tau))
\end{bmatrix},
\quad 0 \leq \tau < d
\]

Now, (16) allows us to write \(J_{\infty}^N\) as

\[
J_{\infty}^N = x^T(0)(\Pi_0^{-1} - \gamma^{-2}P_0)x(0) - \sum_{\tau=0}^{N} \begin{bmatrix}
w(\tau) - \bar{w}(\tau) \\
w(\tau) - \bar{u}(\tau)
\end{bmatrix}^T M_\tau \begin{bmatrix}
w(\tau) - \bar{w}(\tau) \\
w(\tau) - \bar{u}(\tau)
\end{bmatrix} + \sum_{\tau=0}^{N} \begin{bmatrix}
y(\tau) - H^*\bar{x}(\tau)\end{bmatrix}^T R_{\infty}^{-1} \begin{bmatrix}
y(\tau) - H^*\bar{x}(\tau)\end{bmatrix}
\]

In order to achieve an ideal representation, we make a fundamental transformation and then, get the matrix \(M_\tau\) as well as the expected quadratic form as follows:

\[
\hat{M}_\tau = -\gamma^{-2} \begin{bmatrix}
I_p & I_{2m}^T \\
I_p & I_{2m}
\end{bmatrix},
\quad 0 \leq \tau \leq N_d
\]
\[
\hat{M}_\tau = -\gamma^{-2} \begin{bmatrix}
I_p & I_{m}^T \\
I_p & I_{m}
\end{bmatrix},
\quad \tau > N_d
\]

Furthermore, we can use (35) to write (34) as

\[
J_{\infty}^N = x^T(0)(\Pi_0^{-1} - \gamma^{-2}P_0)x(0) + \sum_{\tau=0}^{N} \begin{bmatrix}
w(\tau) - \bar{w}(\tau) \\
w(\tau) - \bar{u}(\tau)
\end{bmatrix}^T S_{\tau} \begin{bmatrix}
w(\tau) - \bar{w}(\tau) \\
w(\tau) - \bar{u}(\tau)
\end{bmatrix} + \begin{bmatrix}
y(\tau) - H^*\bar{x}(\tau)\end{bmatrix}^T \begin{bmatrix}
Q^w_{\tau} & S_{\tau} \\
S_{\tau}^T & Q^u_{\tau}
\end{bmatrix}^{-1} \begin{bmatrix}
y(\tau) - H^*\bar{x}(\tau)
\end{bmatrix}
\]

where

\[
Q^w_{\tau} = \Delta_{\tau}^{-1},
\quad 0 \leq \tau \leq N
\]
\[
S_{\tau} = \begin{bmatrix}
0 & 0 \\
[\bar{S}_{\tau} & 0],
\end{bmatrix},
\quad 0 \leq \tau < d
\]
\[
Q^u_{\tau} = \begin{bmatrix}
diag(\Delta_{\tau}^{-1}, I_\tau),
\quad 0 \leq \tau < d
\end{bmatrix}
\]

Remark 2. Equation (34) deserves such a troublesome transformation since it guarantees that we can find the causal controllers \(u_0(\tau)\) and \(u_1(\tau)\). Otherwise, only the casual \(u_0(\tau)\) can be obtained.

3 Main results

The main results will be stated in the following section.

Proposition 1. If \(M_\tau\) are invertible for any \(\tau\), then, it can be partitioned as

\[
\hat{M}_\tau^{-1} = \begin{bmatrix}
\Delta_\tau^{-1} & \hat{S}_\tau \\
\hat{S}_\tau^T & (\hat{\Delta}_\tau)^{-1}
\end{bmatrix}
\]

where \(\Delta_\tau^{-1}\) is \(p \times p\).

Remark 3. In order to guarantee that every element in performance functions as a lever, \(M_\tau\) tends to be invertible in practice. Therefore, the invertibility of \(M_\tau\) is reasonable.

For a technical reason, we associate the quadratic form (36) with Krein space state-space model as

\[
X(\tau + 1) = (\Phi - G K_{\omega}^*) X(\tau) + GW(\tau) + C(\tau)
\]
\[
U(\tau) = \begin{bmatrix}
K^r \\
H^r
\end{bmatrix} X(\tau) + \begin{bmatrix}
s_\tau(u) \\
0
\end{bmatrix} + V(\tau)
\]

with \(u = u(\cdot)\), where \(u(\cdot)\) is defined in (18), and

\[
C(\tau) = B_0 u_0(\tau) + B_1 u_1(\tau_d) - G[0_m, I_p] \begin{bmatrix}
F_u(0) \\
\end{bmatrix}^T B_1 u_1(k - 1 + \tau_d)
\]

Note that \(C(\tau)\) only has access to the past inputs \(u_1(\tau - 1), \ldots, u_1(\tau - d)\), and the current input \(u_0(\tau)\). The rest of the parameters in (41) \sim (42) are shown as

\[
K_{\omega}^* = -[0_m, I_p][F_u(0)]^T
\]
\[
\hat{K}^r = \begin{bmatrix}
K_0 & 0 \\
K_1 & 0
\end{bmatrix},
\quad 0 \leq \tau \leq N_d
\]
\[
s_\tau(u) = \begin{bmatrix}
\text{col}(s_0, r(u), s_1, r(u)) \\
\text{col}(s_0, s_1, u)
\end{bmatrix},
\quad \tau > N_d
\]
\[
U(\tau) = \begin{bmatrix}
\text{col}(U_0(\tau), U_1(\tau), (13)) \\
\text{col}(U_0(\tau), (13))
\end{bmatrix},
\quad \tau > N_d
\]

In these representations,

\[
K_{\omega}^* = -[I_m, 0_p][F_u(0)]^T
\]
The minimum of $\tau$ in (36) as pseudo-measurements and \( Q_{w_{\tau}}(\tau) \) in (52) are obtained from the projections of \( \mathbf{z}(\tau) \) and \( \mathbf{\tilde{z}}(\tau) \) onto the linear space \( \mathcal{L}\left\{ \left[ \mathbf{u}(i) \right]_{i=0}^{\tau-1} \right\} \), respectively.

3.2 Reorganizing innovations

Observing (50) of the minimal value, we realize that the estimator \( \hat{\mathbf{z}}(\tau|\tau-1) \) and the innovation covariance matrix \( Q_{w_{\tau}}(\tau) \) are the key to design of the controller. At this stage, we face with the problem that the standard Kalman filter formulation is not applicable to computation of \( \hat{\mathbf{z}}(\tau|\tau-1) \) and \( Q_{w_{\tau}}(\tau) \). To conquer it, we shall reorganize the delay-measurements and define reorganized innovation, which, in fact, is the core of the innovation analysis method.

In view of (41) ~ (43), it is not hard to find that

\[
\mathcal{L}\left\{ \left[ \mathbf{u}(i) \right]_{i=0}^{\tau-1} \right\} = \mathcal{L}\left\{ \left[ \mathbf{u}(i) \right]_{i=0}^{\tau-1} \right\}, \quad 0 \leq \tau < d
\]

where

\[
\mathbf{Y}_f(i) = \begin{bmatrix} \mathbf{Y}_0(i) \\ \mathbf{Y}_{1(i)} \end{bmatrix} + \begin{bmatrix} \mathbf{H}_0 \\ \mathbf{H}_1 \end{bmatrix} \mathbf{X}(i) + \mathbf{V}_f(i)
\]

\[
\mathbf{V}_f(i) = \text{col}\{\mathbf{V}_0(i), \mathbf{V}_{1(i)}\}, \quad \mathcal{L}(\mathbf{V}(i)) = I_{2\times2}
\]

\[
\mathcal{L}\left\{ \left[ \mathbf{V}_0(i) \right]_{i=0}^{\tau-1}, \left[ \mathbf{V}_{1(i)} \right]_{i=0}^{\tau-1} \right\} = \Delta_{-1}, \quad 0 \leq i < N_d
\]

\[
\mathcal{L}\left\{ \mathbf{U}_0(i), \mathbf{U}_{1(i)} \right\} = \Delta_{-1}, \quad i > N_d
\]

To avoid confusion, we have to emphasize that just like \( \mathbf{U}(i) \), the orders of \( \Delta_{-1} \) in (57) vary with index \( i \).

Once the delay information is reorganized, it becomes the delay-free measurement so that we can apply it to design the controller directly. \( \mathbf{U}(i), \mathbf{Y}(i) \) are the so-called reorganized measurements. Equations (41), (55), and (43) form a state-space model without delay. Now, let us define the innovation sequence associated with the reorganized measurements.

\[
\mathbf{w}^2(i) = \begin{bmatrix} \mathbf{U}_0(N_{\tilde{i}}) \\ \mathbf{Y}_0(N_{\tilde{i}}) \end{bmatrix} - \begin{bmatrix} \mathbf{U}_0(N_{\tilde{i}} | N_{\tilde{i}} - 1, N_{\tilde{i}}, N_{2\tilde{i}} + i) \\ \mathbf{Y}_0(N_{\tilde{i}} | N_{\tilde{i}} - 1, N_{\tilde{i}}, N_{2\tilde{i}} + i) \end{bmatrix}
\]

\[
\mathbf{w}^1(i) = \begin{bmatrix} \mathbf{U}(\tau_{\tilde{i}}) \\ \mathbf{Y}(\tau_{\tilde{i}}) \end{bmatrix} - \begin{bmatrix} \mathbf{U}(\tau_{\tilde{i}} | \tau_{\tilde{i}} - 1, \tau_{\tilde{i}} - 1, \tau_{2\tilde{i}} + i) \\ \mathbf{Y}(\tau_{\tilde{i}} | \tau_{\tilde{i}} - 1, \tau_{\tilde{i}} - 1, \tau_{2\tilde{i}} + i) \end{bmatrix}
\]
In the following, we will see that although

\[
\begin{aligned}
\mathbf{U}(0|\tau) & = \mathbf{Y}_f(0|\tau), \\
\mathbf{U}(i|\tau) & = \mathbf{Y}_f(i|\tau),
\end{aligned}
\]

with initial estimation value,

\[
\begin{aligned}
\mathbf{U}(0|\tau) & = \mathbf{Y}_f(0|\tau), \\
\mathbf{U}(i|\tau) & = \mathbf{Y}_f(i|\tau),
\end{aligned}
\]

with one-step estimation error,

\[
\begin{aligned}
\mathbf{E}^2(i) & = \mathbf{X}(N_d+i) - \mathbf{X}(N_d+i|N_d+i, N_d, N_{2d}+i) \\
\mathbf{E}^1(i) & = \mathbf{X}(\tau_d+i) - \mathbf{X}(\tau_d+i|\tau_d+i, \tau_d+i, N_d, N_{2d}+i) \\
\mathbf{E}^0(i) & = \mathbf{X}(i) - \mathbf{X}(i|1, i, i-1)
\end{aligned}
\]

In the above equations, \( \mathbf{X}(j|t, s, r) \), \( t \geq s \geq r \), is the estimation of \( \mathbf{X}(j) \) utilizing the measurement sequence

\[
\left\{ \begin{array}{c}
\mathbf{U}(i|t) \\
\mathbf{Y}(i|t)
\end{array} \right\}_{i=t}^{i=t+1} = \left\{ \begin{array}{c}
\mathbf{U}(i|t) \\
\mathbf{Y}(i|t)
\end{array} \right\}_{i=t+1}^{i=t+1}
\]

It is easy to verify that elements in the reorganized innovation \( \{\mathbf{u}(i)\}(i=0, 1, 2) \) are mutually uncorrelated.

### 3.3 Innovation covariance matrix and estimator

In this subsection, we shall provide the general form of the optimal estimator \( \bar{\mathbf{x}}(\tau|t, s, r) \) \( (t \geq s \geq r) \) and the innovation covariance matrix \( Q_{w_s}(\tau) \).

If we denote the covariance matrices of the estimation error \( \mathbf{E}(j) \) as \( P^j_0 \), then, the covariance matrices \( Q_{w_s}(j) \) of the innovation \( \mathbf{u}(j) \) in (58) \( \sim (60) \) can be written as

\[
Q_{w_s}(j) = K_{s-j+N_d}^1 P_{j+N_d-N_s}^2 (K_{j+N_d-N_s})^T + diag(\Delta_{j-N_d}^{-1}, I_s)
\]

\[
Q_{w_s}(j) = K_{s-j+\tau_d}^1 P_{j+\tau_d-N_s}^2 (K_{j+\tau_d-N_s})^T + diag(\Delta_{j+\tau_d-1})^{-1}, I_s)
\]

\[
Q_{w_s}(j) = K_{s-j+1}^1 P_{j}^2 (K_{j})^T + diag(\Delta_{j-1}^{-1})^{-1}, I_s)
\]

In the following, we will see that although \( Q_{w_s}(j) \) depend on \( P_0^j \), they, in turn, help to update \( P_0^j \).

**Lemma 2.** Let \( \Psi = \Phi + G \mathbf{K}_s \). The error covariance matrices \( P_{s-j+N_d}^2, i = 0, 1, 2 \), can be calculated respectively, as

\[
P_{s-j+N_d}^2 = \Psi + \Phi \Psi \Phi^T + G \mathbf{K}_s \mathbf{K}_s^T - \Psi + \Phi \Psi \Phi^T (K_{j+N_d-N_s})^T \times
\]

\[
Q_{s-j+N_d}^{-1} (K_{j+N_d-N_s}) P_{j+N_d-N_s}^2 (K_{j+N_d-N_s})^T + diag(\Delta_{j-N_d}^{-1}, I_s)
\]

\[
P_{s-j+\tau_d}^2 = \Psi + \Phi \Psi \Phi^T + G \mathbf{K}_s \mathbf{K}_s^T - \Psi + \Phi \Psi \Phi^T (K_{j+\tau_d-N_s})^T \times
\]

\[
Q_{s-j+\tau_d}^{-1} (K_{j+\tau_d-N_s}) P_{j+\tau_d-N_s}^2 (K_{j+\tau_d-N_s})^T + diag(\Delta_{j+\tau_d-1})^{-1}, I_s)
\]

\[
P_{s-j+1}^2 = \Psi + \Phi \Psi \Phi^T + G \mathbf{K}_s \mathbf{K}_s^T - \Psi + \Phi \Psi \Phi^T (K_{j})^T \times
\]

\[
Q_{s-j+1}^{-1} (K_{j}) P_{j}^2 (K_{j})^T + diag(\Delta_{j-1}^{-1})^{-1}, I_s)
\]

The proof of Lemma 2 is straightforward and similar to that of [16], thus, it is omitted.

For \( i \geq j \), let

\[
\begin{aligned}
R_{s-j+N_d}^{0,i} & = (X(j|\tau_d), E^0(j|\tau_d)), \quad R_{s-j+\tau_d}^{0,i} = (X(j|\tau_d), E^0(j|\tau_d)) \\
R_{s-j+1}^{0,i} & = (X(j|\tau_d), E^0(j|\tau_d)), \quad R_{s-j+1}^{0,i} = (X(j|\tau_d), E^0(j|\tau_d)) \\
R_{s-j+1}^{2,i} & = (X(j|\tau_d), E^2(j|\tau_d)), \quad R_{s-j+1}^{2,i} = (X(j|\tau_d), E^2(j|\tau_d))
\end{aligned}
\]

be the cross-covariance matrices of the state \( X(j) \) and the state estimation error \( \mathbf{E}^0(i|\tau) \) \( (i = 0, 1, 2) \). Clearly, these cross-covariance matrices can be calculated directly now.

After having made the above preparation, we give the explicit expressions for the innovation covariance matrix \( Q_{w_s}(\tau) \) and the optimal estimator \( \bar{\mathbf{x}}(\tau|\tau_d) \) via a theorem.

**Theorem 1.** Considering the state-space model (41) \( \sim (42) \) in Krein space, the innovation covariance matrix \( Q_{w_s}(\tau) \) is given as

1) For \( 0 \leq \tau < d \),

\[
Q_{w_s}(\tau) = \mathcal{K}_{s+j}^1 \mathbf{Q}_s^0 (\mathcal{K}_{s+j}^1)^T + diag(\Delta_{s-j}^{-1}, I_s)
\]

2) For \( d \leq \tau < N_d \),

\[
Q_{w_s}(\tau) = \mathcal{K}_{s+j+\tau_d}^1 \mathbf{Q}_s^0 (\mathcal{K}_{s+j+\tau_d}^1)^T + \sum_{i=1}^{N_d} R_{s-j+i}^{0,i}(\mathcal{K}_{s-j+i}^1)^T \mathbf{Q}_s(i) \mathcal{K}_{s-j+i}^1 (R_{s-j+i}^{0,i})^T
\]

3) For \( \tau > N_d \),

\[
Q_{w_s}(\tau) = \mathcal{K}_{s+j+1}^1 \mathbf{Q}_s^0 (\mathcal{K}_{s+j+1}^1)^T + \sum_{i=1}^{N_d} R_{s-j+i}^{0,i}(\mathcal{K}_{s-j+i}^1)^T \mathbf{Q}_s(i) \mathcal{K}_{s-j+i}^1 (R_{s-j+i}^{0,i})^T
\]

Meanwhile, the optimal estimator \( \bar{\mathbf{x}}(\tau|\tau_d) = \bar{\mathbf{x}}(\tau|\tau_d) \) \( (l \) is an integer) is updated via the following recursions

1) For \( k > 0 \) and \( N \geq N_{2d+k} + k+1 \geq 0 \),

\[
\bar{\mathbf{x}}(N_d+k+1|N_d+k, N_d, N_{2d+k}+k+1) = \Psi_{N_{2d+k}} \bar{\mathbf{x}}(N_d+k|N_d+k-1, N_d, N_{2d+k}+k) +
\]

\[
C(N_d+k) + \Psi_{N_{2d+k}} R_{N_{2d+k}}^2 \mathcal{K}_{N_{2d+k}}^1 (\mathcal{K}_{N_{2d+k}}^1)^T Q_{w_s}^{-1} \mathbf{w}^2(1)
\]

2) For \( k = 0 \) and \( N \geq N_{2d+k} + k+1 \)

\[
\bar{\mathbf{x}}(N_{2d+k}+1|N_d+k, N_d, N_{2d+k}+k+1) = \Psi_{N_{2d+k}} \bar{\mathbf{x}}(N_{2d+k}+1|N_d+k, N_d, N_{2d+k}+k) +
\]

\[
C(N_{2d+k}) + \Psi_{N_{2d+k}} R_{N_{2d+k}}^2 \mathcal{K}_{N_{2d+k}}^1 (\mathcal{K}_{N_{2d+k}}^1)^T Q_{w_s}^{-1} \mathbf{w}^2(1)
\]
2) For $k > 0$ and $N_d \geq \tau_{2d} + k + 1 \geq 0$,
\begin{equation}
\dot{x}(\tau_d + k + 1|\tau_d, \tau_d + k, \tau_d + 2d + k + 1) = \Psi_{\tau_d + k} \dot{x}(\tau_d + k|\tau_d + k - 1, \tau_d + k - 1, \tau_{2d} + k) + C(\tau_d + k) + \hat{R}_{h, k}(K_{h, k}^{-1})^T Q_{w, h}^{-1} \hat{w}(k)
\end{equation}
(73)

\begin{equation}
\dot{x}(\tau_d + 1|\tau_d, \tau_d) = \Psi_{\tau_d} \dot{x}(\tau_d|1, \tau_d - 1, \tau_d - 1) + C(\tau_d) + \hat{R}_{0, 0}(K_{0, 0}^{-1})^T Q_{w, 0}^{-1} \hat{w}(\tau_d)
\end{equation}
(74)

\begin{equation}
\dot{x}(\tau_d + k + 1|\tau_d + k, \tau_d + k, \tau_d + 2d + k + 1) = \dot{x}(\tau_d + k + 1|\tau_d + k, \tau_d + k, \tau_d + 2d + k) + \hat{R}_{\tau_d + k + 1, k + 1}(K_{\tau_d + k + 1, k + 1}^{-1})^T Q_{w, \tau_d + k + 1}^{-1} \hat{w}(\tau_d + 2d + k + 1) + \sum_{i=1}^{d-1} \hat{R}_{\tau_d + k + 1, i}(K_{\tau_d + k + 1, i}^{-1})^T Q_{w, \tau_d + k + 1}^{-1} \hat{w}(i)
\end{equation}
(75)

3) one-step predict formula is shown as
\begin{equation}
\dot{x}(\tau + 1|\tau, \tau, \tau) = \Psi_{\tau} \dot{x}(\tau|\tau - 1, \tau - 1, \tau - 1) + C(\tau) + \Psi_{\tau} \hat{r}_{\tau, \tau}^0 Q_{w, \tau}^{-1} \hat{w}(\tau)
\end{equation}
(76)

with initial value $\dot{x}(0|0, -1, -1, -1) = 0$.
For a simple expression of the optimal estimator, we omit the time index of innovation covariance matrix, which is identical with that of the adjacent innovation when it does not cause confusion.

3.4 Solutions for the $H_\infty$ measurement control problem

By the virtue of the discussions in the previous sections, we formulate the $H_\infty$ measurement-feedback control law and the sufficient and necessary condition thereof.

**Theorem 2.** Consider the state-space model (1) $\sim$ (4).

For a given $\gamma > 0$, a measurement-feedback $H_\infty$ controller $u_i(t) = F_i(y(j)|0 \leq j \leq 1) (i = 0, 1)$ that makes (9) holds exist if and only if

1) $H_\infty > 0$; $\gamma > 2$;

2) $\Delta_x > 0$ for all $\tau = 0, 1, \cdots, N$;

3) The matrix $Q_{w, \tau} - S_{\tau}(Q_{w, \tau}^{-1})^T S_{\tau}$ has the same inertia as $Q_{w, \tau}$ for all $\tau = 0, 1, \cdots, N$.

Moreover, for $\tau \leq N_d$, the central controller is given by
\begin{align}
\dot{u}_0(\tau) &= K_0 \dot{x}(\tau|\tau - 1) + [I_m \ 0_m] Q_{w, \tau} Q_{w, \tau}^{-1} (y(\tau) - H^T \dot{x}(\tau|\tau - 1)) + s_{0, v}(u) \\
\dot{u}_1(\tau) &= K_1 \dot{x}(\tau|\tau - 1) + [0_m \ I_m] Q_{w, \tau} Q_{w, \tau}^{-1} (y(\tau) - H^T \dot{x}(\tau|\tau - 1)) + s_{1, v}(u)
\end{align}

For $\tau > N_d$, the central controller is given by
\begin{align}
\dot{u}_0(\tau) &= K_0 \dot{x}(\tau|\tau - 1) + Q_{w, \tau} Q_{w, \tau}^{-1} (y(\tau) - H^T \dot{x}(\tau|\tau - 1)) + s_{0, v}(u) \\
\dot{u}_1(\tau) &= K_1 \dot{x}(\tau|\tau - 1) + Q_{w, \tau} Q_{w, \tau}^{-1} (y(\tau) - H^T \dot{x}(\tau|\tau - 1)) + s_{1, v}(u)
\end{align}

where
\begin{equation}
Q_{w, \tau} = \begin{cases}
(K_0^T)^T (K_1^T)^T P_{0, \tau} H_{0, \tau}^T, & 0 \leq \tau < d \\
(K_0 P_{0, \tau} H_{0, \tau}^T K_0 R_{0, \tau}^0 H_{0, \tau}^T)^T + I_{2v}, & d \leq \tau \leq N_d \\
(K_1 P_{1, \tau} H_{1, \tau}^T K_1 R_{1, \tau}^0 H_{1, \tau}^T)^T + I_{2v}, & \tau > N_d
\end{cases}
\end{equation}

\begin{equation}
Q_y = \begin{cases}
H_0 P_{0, \tau} H_{0, \tau}^T + I_{3v}, & 0 \leq \tau < d \\
H_0 P_{0, \tau} H_{0, \tau}^T + H_0 R_{0, \tau}^0 H_{0, \tau}^T + I_{2v}, & d \leq \tau \leq N_d \\
H_0 P_{0, \tau} H_{0, \tau}^T + H_0 R_{0, \tau}^0 H_{0, \tau}^T + I_{2v}, & \tau > N_d
\end{cases}
\end{equation}

**Proof.** The sufficient and necessary condition of the existence of the $H_\infty$ measurement-feedback controller $u_i(t) = F_i(y(j)|0 \leq j \leq 1) (i = 0, 1)$ can be referred to [17] directly. As for the central controller, it will be clear after we make an LDU decomposition for $Q_{w, \tau}$ in (50).

Different from the delay-free case, the present controller involves something else besides the state-estimation, which originates from the delay-input joining the original state equation. Actually, the $H_\infty$ state-feedback controller [16] has already included the additional term besides the state.

Towards the end, let us analyze the relationship between [4, 16] and the paper. In fact, three of them share similar ideas to some extent.

**Remark 4.** Reference [4] is clearly a special case of our paper, so is its result. In addition, when the perfect and uncontaminated states can be observed directly, namely, the estimator of states is accurate, the result of [16] is also a special case of this paper.

This paper achieves the causal and central $H_\infty$ controller of the time-invariant system with single I/O delay. Especially, the idea can be extended to the time-varying or multi-delay systems.

**Remark 5.** The present approach can be extended to deal with the multi-delay and vary-time case trivially.

4 Numerical example

In order to display that the present controller is effective, we still take the model in [18] with respect to the network congestion control and follow almost all of Zhang's [16] parameters.

Generally speaking, at time instant $t$, the high priority sources $\xi_1, \xi_{r, h, 1}$, and $\xi_{r, h, -1}$, and the queue lengths $q$ and $q_{r, h}$, can be obtained reliably. Therefore, we can introduce measurement equations, such as
\begin{equation}
y_0(t) = H_0 x(t) + v_0(t)
\end{equation}
(77)
\begin{equation}
y_1(t) = H_1 x(t - h_1) + v_1(t)
\end{equation}
(78)
with the single delay $h_1 = d = 5$, $H_0 = [0 \ 1 \ 0]$, and $H_1 = [1 \ 0 \ 0]$.

For a prescribed $\gamma > 0$, the $H_\infty$ congestion control with measurement-feedback desires to find source rate $v_{i, h}$ so that
\begin{equation}
\sup_{\{q_0, v_{i, h}, w(\cdot), v_0(\cdot), v_1(\cdot)\}} J(q_0, v_{i, h}, w(\cdot), v_0(\cdot), v_1(\cdot)) < \gamma^2
\end{equation}
(79)
where
\begin{equation}
J(q_0, v_{i, h}, w(\cdot), v_0(\cdot), v_1(\cdot)) = \sum_{i=0}^{N} \left[ (q_i - q_{i+1})^2 + (v_{i, h} - \mu)^2 \right] / \sum_{i=0}^{N} (w_i(t)^2 + v_0(t)^2 + v_1(t)^2)
\end{equation}
(80)

In practice, the performance (79) aims to keep the queue in the buffer close to the target length $q$ and the source rate close to the nominal service rate $\mu$, i.e., control input $u_1$ in the neighborhood of 0.

The simulation result with respect to Theorem 2 can be seen in Figs. 1 and 2. It is shown that the controller is effective. Moreover, the desired performance is achieved.
5 Conclusion

The paper provides a solution to the $H_\infty$ measurement-feedback control problem for systems with single input-delay and measurement-delay via introducing pseudo-measurements as well as Krein space. It is testified that the reorganizing technique is very effective to conquer the difficulty aroused by delays. The results also show us that the solution has almost the same structure as the full-information control-law, where the states are replaced by their optimal estimations now. Furthermore, we see that the present separation principle is slightly different from the one first presented in [19], and the former is more natural the present separation principle is slightly different from the one first presented in [19], and the former is more natural.

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