Stability Analysis on Predictive Control of Discrete-Time Systems with Input Nonlinearity[1)]

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Abstract For systems with input nonlinearities, a two-step control scheme is adopted. For linear part the control law with Riccati iteration matrices satisfying certain conditions is used to get the Lyapunov function. The stability conditions are investigated, considering the reversion errors coming from solving nonlinear algebraic equation and desaturation computation, which give tuning guidelines for the real systems. Simulation studies validate the results of theoretical analysis.

Key words Input nonlinearity, two-step control, predictive control, Riccati iteration, stability

1 Introduction

In process industries, many systems have input nonlinearities such as saturation, dead time, relay cycle, *etc.* Moreover, chemical processes represented by Hammerstein model in the form of "static nonlinear+dynamic linear", such as pH neutralization, high purity distillation, *etc.*, can also be taken as input nonlinear systems. Generally, two-step control can be applied to this kind of systems[1~3], in which a desired intermediate variable is firstly obtained by applying linear model and then the real control action is obtained by solving nonlinear algebraic equation group(NAEG), desaturation, *etc.* The advantage of two-step control is that the controller design is still within the scope of linear systems, which is much simpler than nonlinear control incorporating the nonlinearities into system equation or objective function.

For an input nonlinear control system designed by two-step scheme, if the real control input recurs desired intermediate variable exactly through static nonlinearities, the stability of the system could be guaranteed by properly designing the linear system. However, it is difficult to meet this perfect condition in real applications. For the Hammerstein system, solving NAEG will inevitably have error, and for the input saturated system, the restricted input is often largely different from the desired one. In these cases, the stability analysis of two-step controller becomes very difficult.

It is the aim of this paper to study the stability property of two step model predictive control systems with input nonlinearities (TSMPC), including Hammerstein nonlinearity, input saturation, *etc.* Applying Lyapunov's stability theory, we obtain some stability conclusions of this kind of systems. Section 2 describes the main idea of TSMPC. Section 3 gives the stability conditions. Section 4 illustrates the stability tuning of TSMPC and Section 5 gives a simulation example.

2 The description of TSMPC

Consider the following discrete-time system with input nonlinearity,

$$\boldsymbol{z}_{k+1} = A\boldsymbol{z}_k + B\boldsymbol{x}_k, \quad \boldsymbol{y}_k = C\boldsymbol{z}_k, \quad \boldsymbol{x}_k = \boldsymbol{\phi}(\boldsymbol{u}_k) \tag{1}$$

where $\boldsymbol{z}\in R^n, \boldsymbol{x}\in R^m, \boldsymbol{y}\in R^p, \boldsymbol{u}\in R^m$ are state, intermediate variable, output and input, respectively. $\boldsymbol{\phi}$ represents the relationship between input and the intermediate variable satis-

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fying $\boldsymbol{\phi}(0)=0$. (A,B) is assumed stabilizable. We also assume that $\boldsymbol{\phi}$ includes Hammerstein static nonlinearity $\boldsymbol{f}$ and input saturation constraints $sat_i, i=1,\cdots,m$.

For the above nonlinear system, the structure of general two-step controller is shown in Fig. 1. In the following we describe the concrete realization of TSMPC.

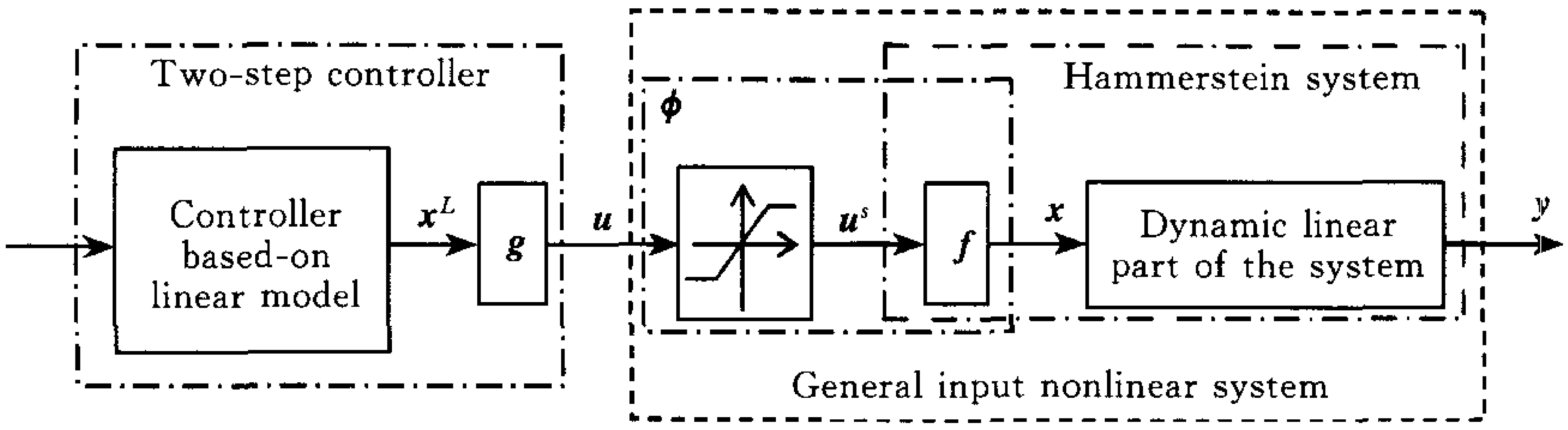

Fig. 1 The structure of controller and system cascade form

In the first step, we only consider the linear system $z_{k+1}=Az_k+Bx_k, y_k=Cz_k$ and define the objective function as

$$J(N,x_k) = \sum_{j=0}^{N-1}[z_{k+j}^{\mathrm{T}}Qz_{k+j} + x_{k+j}^{\mathrm{T}}Rx_{k+j}] + z_{k+N}^{\mathrm{T}}Q_N z_{k+N} \tag{2}$$

where $Q=Q^{\mathrm{T}}\geqslant 0$ and $R>0$ are weighting matrices of the state and the intermediate variable; $Q_N\geqslant 0$ is the terminal state weighting matrix and $Q\neq Q_N$ is the usual case in predictive control. The following Riccati iteration is adopted:

$$P_j = Q + A^{\mathrm{T}}P_{j+1}A - A^{\mathrm{T}}P_{j+1}B(R+B^{\mathrm{T}}P_{j+1}B)^{-1}B^{\mathrm{T}}P_{j+1}A, \quad j<N,\ P_N = Q_N \tag{3}$$

to obtain the linear control law:

$$x_k = -(R+B^{\mathrm{T}}P_1B)^{-1}B^{\mathrm{T}}P_1Az_k \tag{4}$$

Note that x_k in (4) may be unable to be implemented by real control action, so we denote it as

$$x_k^L = Kz_k = -(R+B^{\mathrm{T}}P_1B)^{-1}B^{\mathrm{T}}P_1Az_k \tag{5}$$

In the second step, we solve $x_k^L - \boldsymbol{f}(\hat{u}_k)=0$ to obtain $\hat{u}_k=\boldsymbol{\varphi}(x_k^L)$, then obtain u_k by desaturation, $u_k=sat\{\hat{u}_k\}$, which is formalized as $u_k=\boldsymbol{g}(x_k^L)$ as shown in Fig. 1. Note that for one x_k^L several $\hat{u}_k$ as well as u_k may be obtained. However, adding extra conditions (such as choosing u_k to be the closest to u_{k-1}, choosing u_k with smallest amplitude, *etc.*) we can obtain a most suitable u_k. When u_k has been implemented, the corresponding x_k is denoted as $x_k=\boldsymbol{f}(sat\{u_k\})=\boldsymbol{f}(sat\{\boldsymbol{g}(x_k^L)\})=(\boldsymbol{\phi}\cdot\boldsymbol{g})(x_k^L)=\boldsymbol{h}(x_k^L)$. Then, the control law of TSMPC represented by the intermediate variable becomes

$$x_k = \boldsymbol{h}(x_k^L) = \boldsymbol{h}(-(R+B^{\mathrm{T}}P_1B)^{-1}B^{\mathrm{T}}P_1Az_k) \tag{6}$$

With (6) we obtain the closed-loop system

$$z_{k+1} = Az_k + Bx_k = (A - B(R+B^{\mathrm{T}}P_1B)^{-1}B^{\mathrm{T}}P_1A)z_k + B(\boldsymbol{h}(x_k^L) - x_k^L) \tag{7}$$

The closed-loop structure of TSMPC is shown in Fig. 2.

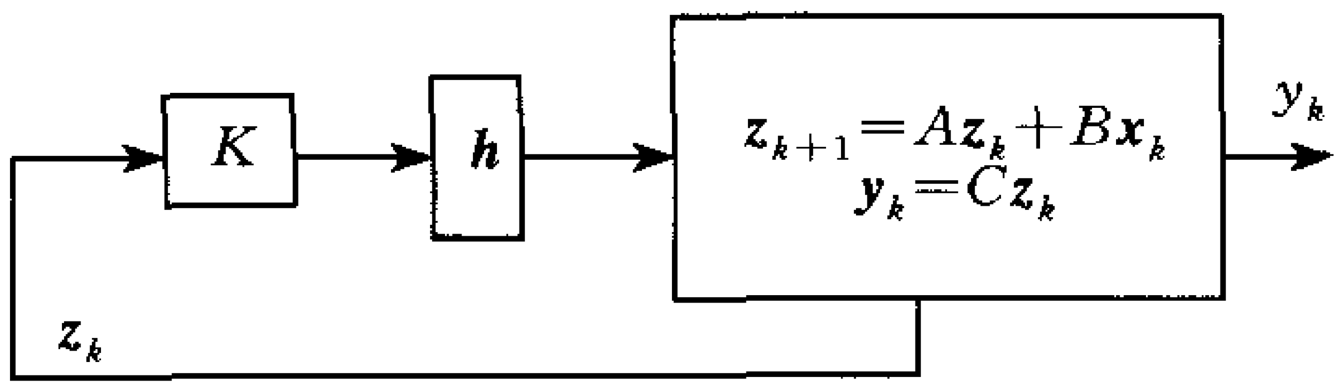

Fig. 2 The closed-loop structure of TSMPC

When $\boldsymbol{h}=\boldsymbol{1}=[1,1,\cdots 1]^{\mathrm{T}}$, the nonlinear item in (7) disappears. However, generally $\boldsymbol{h}=\boldsymbol{1}$ can not hold, since in the actual solution of TSMPC, $\boldsymbol{h}$ may incorporate:

i) the solution error of NAEG;

ii) the desaturation action that makes $x_k^L\neq x_k$.

Moreover, for a real system $\boldsymbol{h}$ may also incorporate:

iii) the modeling error of the Hammerstein nonlinearity;

iv) the uncertainty in the actuator and the execution error[4].

Hence, in real applications $\boldsymbol{h}\neq 1$ (i.e., $x_k^L\neq x_k$) will be always true. It is just this reason that brings the complexity of the stability analysis for TSMPC. In the next chapter we adopt Lyapunov's theory to analyze the stability of TSMPC.

3 Stability analysis of TSMPC

In the following, we choose $R=\lambda I$ for convenience.

Theorem 1. For system represented by (1), TSMPC is adopted. Then under the following two conditions the closed-loop system is exponentially stable:

i) The control parameters $\{Q_N, Q, \lambda, N\}$ satisfy $Q>P_0-P_1$

ii) The nonlinearity $\boldsymbol{h}$ satisfies

$$-\lambda \boldsymbol{s}^{\mathrm{T}}[2\boldsymbol{h}(\boldsymbol{s})-\boldsymbol{s}]+(\boldsymbol{h}(\boldsymbol{s})-\boldsymbol{s})^{\mathrm{T}}B^{\mathrm{T}}P_1B(\boldsymbol{h}(\boldsymbol{s})-\boldsymbol{s})\leqslant 0 \tag{8}$$

Proof. Define Lyapunov function as $V(\boldsymbol{z}_k)=\boldsymbol{z}_k^{\mathrm{T}}P_1\boldsymbol{z}_k$. Then

$$\begin{aligned}
&V(\boldsymbol{z}_{k+1})-V(\boldsymbol{z}_k)=\\
&\boldsymbol{z}_k^{\mathrm{T}}[A-B(\lambda I+B^{\mathrm{T}}P_1B)^{-1}B^{\mathrm{T}}P_1A]^{\mathrm{T}}P_1[A-B(\lambda I+B^{\mathrm{T}}P_1B)^{-1}B^{\mathrm{T}}P_1A]\boldsymbol{z}_k-\boldsymbol{z}_k^{\mathrm{T}}P_1\boldsymbol{z}_k+\\
&2\lambda \boldsymbol{z}_k^{\mathrm{T}}A^{\mathrm{T}}P_1B(\lambda I+B^{\mathrm{T}}P_1B)^{-1}(\boldsymbol{h}(\boldsymbol{x}_k^L)-\boldsymbol{x}_k^L)+(\boldsymbol{h}(\boldsymbol{x}_k^L)-\boldsymbol{x}_k^L)^{\mathrm{T}}B^{\mathrm{T}}P_1B(\boldsymbol{h}(\boldsymbol{x}_k^L)-\boldsymbol{x}_k^L)=\\
&\boldsymbol{z}_k^{\mathrm{T}}[-Q+P_0-P_1-A^{\mathrm{T}}P_1B(\lambda I+B^{\mathrm{T}}P_1B)^{-1}\lambda(\lambda I+B^{\mathrm{T}}P_1B)^{-1}B^{\mathrm{T}}P_1A]\boldsymbol{z}_k+\\
&2\lambda \boldsymbol{z}_k^{\mathrm{T}}A^{\mathrm{T}}P_1B(\lambda I+B^{\mathrm{T}}P_1B)^{-1}(\boldsymbol{h}(\boldsymbol{x}_k^L)-\boldsymbol{x}_k^L)+(\boldsymbol{h}(\boldsymbol{x}_k^L)-\boldsymbol{x}_k^L)^{\mathrm{T}}B^{\mathrm{T}}P_1B(\boldsymbol{h}(\boldsymbol{x}_k^L)-\boldsymbol{x}_k^L)=\\
&\boldsymbol{z}_k^{\mathrm{T}}(-Q+P_0-P_1)\boldsymbol{z}_k-\lambda(\boldsymbol{x}_k^L)^{\mathrm{T}}\boldsymbol{x}_k^L-2\lambda(\boldsymbol{x}_k^L)^{\mathrm{T}}(\boldsymbol{h}(\boldsymbol{x}_k^L)-\boldsymbol{x}_k^L)+\\
&(\boldsymbol{h}(\boldsymbol{x}_k^L)-\boldsymbol{x}_k^L)^{\mathrm{T}}B^{\mathrm{T}}P_1B(\boldsymbol{h}(\boldsymbol{x}_k^L)-\boldsymbol{x}_k^L)=\boldsymbol{z}_k^{\mathrm{T}}(-Q+P_0-P_1)\boldsymbol{z}_k-\lambda(\boldsymbol{x}_k^L)^{\mathrm{T}}(2\boldsymbol{h}(\boldsymbol{x}_k^L)-\boldsymbol{x}_k^L)+\\
&(\boldsymbol{h}(\boldsymbol{x}_k^L)-\boldsymbol{x}_k^L)^{\mathrm{T}}B^{\mathrm{T}}P_1B(\boldsymbol{h}(\boldsymbol{x}_k^L)-\boldsymbol{x}_k^L)
\end{aligned}$$

Apparently the system will be stable under the conditions i)~ii). Moreover, since $Q>P_0-P_1$, $V(\boldsymbol{z}_{k+1})-V(\boldsymbol{z}_k)\leqslant-\lambda_{\min}(Q-P_0+P_1)\|\boldsymbol{z}_k\|<0$, $\forall\, \boldsymbol{z}_k\neq 0$ and "exponentially" holds. □

Theorem 1 embodies the characteristic of two-step control, where condition i) is a stability requirement on the linear control law while condition ii) on the nonlinearity $\boldsymbol{h}$. From the proof of Theorem 1 we can easily know that condition i) is a sufficient stability condition of the linear control law and can be satisfied by properly choosing the control parameters. Moreover, if there is not reversion error, then $\boldsymbol{h}(s)=s$ and (8) becomes $-\lambda \boldsymbol{s}^{\mathrm{T}}\boldsymbol{s}\leqslant 0$ which will be always true, and by Theorem 1 we obtain the stability condition for linear control law.

With condition i) satisfied, we can further investigate the stability requirement on $\boldsymbol{h}$ under condition ii). To do this we assume

$$\|\boldsymbol{h}(\boldsymbol{s})\|\geqslant b_1\|\boldsymbol{s}\|, \quad \|\boldsymbol{h}(\boldsymbol{s})-\boldsymbol{s}\|\leqslant|b-1|\cdot\|\boldsymbol{s}\| \tag{9}$$

where $b>0$ and $b_1>0$ are scalars; $\|v\|$ stands for 2-norm of vector v; $\|\boldsymbol{h}(\boldsymbol{s})\|\geqslant b_1\|s\|$ mainly stands for the requirement on desaturation level while $\|\boldsymbol{h}(\boldsymbol{s})-\boldsymbol{s}\|\leqslant|b-1|\,\|s\|$ embodies the restriction on the total reversion error. Now we can obtain the following requirement on the reversion error.

Corollary 1. For system represented by (1), assume that the control law (4) satisfies $Q>P_0-P_1$. Then the control law (6) will exponentially stabilize the system if b, b_1 in (9) satisfy the following requirement:

$$-\lambda[b_1^2-(b-1)^2]+(b-1)^2\sigma_{\max}(B^{\mathrm{T}}P_1B)\leqslant 0 \tag{10}$$

Proof. Applying (9) we make following deductions:

$$\begin{aligned}
&-\lambda \boldsymbol{s}^{\mathrm{T}}[2\boldsymbol{h}(\boldsymbol{s})-\boldsymbol{s}]+(\boldsymbol{h}(\boldsymbol{s})-\boldsymbol{s})^{\mathrm{T}}B^{\mathrm{T}}P_1B(\boldsymbol{h}(\boldsymbol{s})-\boldsymbol{s})=\\
&-\lambda \boldsymbol{h}(\boldsymbol{s})^{\mathrm{T}}\boldsymbol{h}(\boldsymbol{s})+(\boldsymbol{h}(\boldsymbol{s})-\boldsymbol{s})^{\mathrm{T}}(\lambda I+B^{\mathrm{T}}P_1B)(\boldsymbol{h}(\boldsymbol{s})-\boldsymbol{s})\leqslant\\
&-\lambda b_1^2\boldsymbol{s}^{\mathrm{T}}\boldsymbol{s}+(\boldsymbol{h}(\boldsymbol{s})-\boldsymbol{s})^{\mathrm{T}}(\lambda I+B^{\mathrm{T}}P_1B)(\boldsymbol{h}(\boldsymbol{s})-\boldsymbol{s})\leqslant\\
&-\lambda b_1^2\boldsymbol{s}^{\mathrm{T}}\boldsymbol{s}+(b-1)^2\sigma_{\max}(\lambda I+B^{\mathrm{T}}P_1B)\boldsymbol{s}^{\mathrm{T}}\boldsymbol{s}=\\
&-\lambda b_1^2\boldsymbol{s}^{\mathrm{T}}\boldsymbol{s}+\lambda(b-1)^2\boldsymbol{s}^{\mathrm{T}}\boldsymbol{s}+(b-1)^2\sigma_{\max}(B^{\mathrm{T}}P_1B)\boldsymbol{s}^{\mathrm{T}}\boldsymbol{s}=\\
&-\lambda[b_1^2-(b-1)^2]\boldsymbol{s}^{\mathrm{T}}\boldsymbol{s}+(b-1)^2\sigma_{\max}(B^{\mathrm{T}}P_1B)\boldsymbol{s}^{\mathrm{T}}\boldsymbol{s}
\end{aligned}$$

Hence, if condition (10) holds, (8) can be deduced from (9). Therefore, the corollary holds. □

In real applications, f is often chosen as totally decoupled form. This simplifies not only the NAEG solving but also f identifying. In the case f is totally decoupled, we can assume

$$b_{i,1}s_i^2 \leqslant h_i(s_i)s_i \leqslant b_{i,2}s_i^2, \quad i=1,\cdots,m \tag{11}$$

where $b_{i,2} \geqslant b_{i,1} > 0$ are scalars. Apparently (11) has clearer meaning than (9).

Since $h_i(s_i)$ has the same sign with s_i, $|h_i(s_i)-s_i| = ||h_i(s_i)|-|s_i|| \leqslant \max\{|b_{i,1}-1|, |b_{i,2}-1|\} \cdot |s_i|$. Let $b_1 = \min\{b_{1,1}, b_{2,1}, \cdots, b_{m,1}\}$ and $|b-1| = \max\{|b_{1,1}-1|, \cdots, |b_{m,1}-1|, |b_{1,2}-1|, \cdots, |b_{m,2}-1|\}$; then (9) can be deduced from (11), so Corollary 1 will still hold for (11). Moreover, we can obtain the following conclusion.

Corollary 2. For system represented by (1), assume that f has totally decoupled form and the control law (4) satisfies $Q > P_0 - P_1$. Then the control law (6) will exponentially stabilize the system if the following condition holds:

$$-\lambda(2b_1-1) + (b-1)^2\sigma_{\max}(B^{\mathrm{T}}P_1B) \leqslant 0 \tag{12}$$

Proof. By using (11), it is easy to conclude that $s_i[h_i(s_i)-s_i] \geqslant s_i[b_{i,1}s_i - s_i]$, $i=1, \cdots, m$. Since

$$\begin{aligned}
&-\lambda \boldsymbol{s}^{\mathrm{T}}[2\boldsymbol{h}(\boldsymbol{s}) - \boldsymbol{s}] + (\boldsymbol{h}(\boldsymbol{s}) - \boldsymbol{s})^{\mathrm{T}}B^{\mathrm{T}}P_1B(\boldsymbol{h}(\boldsymbol{s}) - \boldsymbol{s}) = \\
&-\lambda \boldsymbol{s}^{\mathrm{T}}\boldsymbol{s} - 2\lambda \boldsymbol{s}^{\mathrm{T}}[\boldsymbol{h}(\boldsymbol{s}) - \boldsymbol{s}] + (\boldsymbol{h}(\boldsymbol{s}) - \boldsymbol{s})^{\mathrm{T}}B^{\mathrm{T}}P_1B(\boldsymbol{h}(\boldsymbol{s}) - \boldsymbol{s}) \leqslant \\
&-\lambda \boldsymbol{s}^{\mathrm{T}}\boldsymbol{s} - 2\lambda \sum_{i=1}^{m}(b_{i,1}-1)s_i^2 + (\boldsymbol{h}(\boldsymbol{s}) - \boldsymbol{s})^{\mathrm{T}}B^{\mathrm{T}}P_1B(\boldsymbol{h}(\boldsymbol{s}) - \boldsymbol{s}) \leqslant \\
&-\lambda \boldsymbol{s}^{\mathrm{T}}\boldsymbol{s} - 2\lambda(b_1-1)\boldsymbol{s}^{\mathrm{T}}\boldsymbol{s} + (b-1)^2\sigma_{\max}(B^{\mathrm{T}}P_1B)\boldsymbol{s}^{\mathrm{T}}\boldsymbol{s} = \\
&-\lambda(2b_1-1)\boldsymbol{s}^{\mathrm{T}}\boldsymbol{s} + (b-1)^2\sigma_{\max}(B^{\mathrm{T}}P_1B)\boldsymbol{s}^{\mathrm{T}}\boldsymbol{s}
\end{aligned}$$

if condition (12) is satisfied, then (11) satisfies (8). Therefore, the corollary holds. □

Remark 1. If $f=1$, that is, if there is only input saturation nonlinearity, then $b_2=1$, $(b-1)^2=(b_1-1)^2$, and both conditions (10) and (12) will have the form $-\lambda(2b_1-1)+(b_1-1)^2\sigma_{\max}(B^{\mathrm{T}}P_1B) \leqslant 0$.

4 Stability tuning of TSMPC

4.1 Evaluating the bounds of *h* for a real system

According to (10) and (12), we can give the tuning guideline for TSMPC, that is, if the controlled system is not stable yet, we can tune Q_N, Q, λ and N to restabilize it. To do this we should first determine b_1 and $|b-1|$. In the following we illustrate how to evaluate b_1 and $|b-1|$ for a single input system. Denote

$$sat\{\hat{u}\} = \begin{cases} u_{s,\min}, & \hat{u} \leqslant u_{s,\min} \\ \hat{u}, & u_{s,\min} \leqslant \hat{u} \leqslant u_{s,\max} \\ u_{s,\max}, & \hat{u} \geqslant u_{s,\max} \end{cases} \tag{13}$$

Assume that:

A) The solution error of $x^L = f(\hat{u})$ is restricted to $\underline{b}(x^L)^2 \leqslant f \cdot \varphi(x^L)x^L \leqslant \bar{b}(x^L)^2$ where $\underline{b} > 0$ and $\bar{b} > 0$ are constants;

B) The design of TSMPC satisfies $x^L_{\min} \leqslant x^L \leqslant x^L_{\max}$ and $x^L = f(\hat{u})$ always has real-valued solution;

C) Let $x_{s,\min} = f(u_{s,\min})$ and $x_{s,\max} = f(u_{s,\max})$; then $x^L_{\min} \leqslant x_{s,\min} < 0$ and $0 < x_{s,\max} \leqslant x^L_{\max}$;

D) If $\varphi(x^L) \leqslant u_{s,\min}$, then $x^L \leqslant x_{s,\min}$, and if $\varphi(x^L) \geqslant u_{s,\max}$, then $x^L \geqslant x_{s,\max}$.

Denote $b_s = \min\{x_{s,\min}/x^L_{\min}, x_{s,\max}/x^L_{\max}\}$. It is easy to conclude that

$$h(x^L) = f \cdot sat \cdot g(x^L) = f \cdot sat \cdot \varphi(x^L) =$$

$$\begin{cases} f(u_{s,\min}), & \varphi(x^L) \leqslant u_{s,\min} \\ f\varphi(x^L), & u_{s,\min} \leqslant \varphi(x^L) \leqslant u_{s,\max} \\ f(u_{s,\max}), & \varphi(x^L) \geqslant u_{s,\max} \end{cases} = \begin{cases} x_{s,\min}, & \varphi(x^L) \leqslant u_{s,\min} \\ f\varphi(x^L), & u_{s,\min} \leqslant \varphi(x^L) \leqslant u_{s,\max} \\ x_{s,\max}, & \varphi(x^L) \geqslant u_{s,\max} \end{cases} \tag{14}$$

Evaluate the bounds of the three cases in (14) we obtain

$$\begin{cases} b_s(x^L)^2 \leqslant x_{s,\min}x^L \leqslant (x^L)^2, & \varphi(x^L) \leqslant u_{s,\min} \\ \underline{b}(x^L)^2 \leqslant f\varphi(x^L)x^L \leqslant \bar{b}(x^L)^2, & u_{s,\min} \leqslant \varphi(x^L) \leqslant u_{s,\max} \\ b_s(x^L)^2 \leqslant x_{s,\max}x^L \leqslant (x^L)^2, & \varphi(x^L) \geqslant u_{s,\max} \end{cases} \tag{15}$$

Integrating (14) and (15) we obtain

$$\min\{b_s,\underline{b}\}(x^L)^2 \leqslant h(x^L)x^L \leqslant \max\{1,\bar{b}\}(x^L)^2 \tag{16}$$

Hence, $b_1=\min\{b_s,\underline{b}\}$, $b_2=\max\{1, \bar{b}\}$ and $|b-1|=\max\{|b_1-1|,|b_2-1|\}$

Note that the above assumptions (A)~(B) are fundamental conditions for evaluating the bounds of $\boldsymbol{h}$, while (C)~(D) restrict our discussion to concrete nonlinearity but without loss of generality. In other cases, the bounds of $\boldsymbol{h}$ can be obtained similarly. In real applications, the bounds of $\boldsymbol{h}$ can be evaluated based on the concrete situations.

4.2 Remarks on parameter tuning for real system

Denote (10) and (12) as

$$\sigma_{\max}(B^{\mathrm{T}}P_1B)/\lambda \leqslant [b_1^2-(b-1)^2]/(b-1)^2 \tag{17}$$

and

$$\sigma_{\max}(B^{\mathrm{T}}P_1B)/\lambda \leqslant (2b_1-1)/(b-1)^2 \tag{18}$$

Unify (17)~(18) as

$$\sigma_{\max}(B^{\mathrm{T}}P_1B)/\lambda \leqslant \delta(b,b_1) \tag{19}$$

Apparently, reducing the solution error to increase $\delta(b,b_1)$ will be advantageous for stabilization. But the effect of this way is limited. According to (19) we can give the following control parameter tuning guideline.

Step1. If the system is stable, go to step 5.

Step2. Evaluate $\delta(b,b_1)$. Since b_1 may change with λ,Q_N,Q and N, $\delta(b,b_1)$ should be reevaluated after each tuning of $\{\lambda,Q_N,Q,N\}$.

Step3. Check if condition (19) is satisfied. If it isn't, tune λ to satisfy it.

Step4. Check if $Q>P_0-P_1$. If it is, go to step5, else tune $\{Q_N,Q,N\}$ to satisfy it and go to step2. If (19) and $Q>P_0-P_1$ can not be satisfied by repeated tunings, (19) needs only to be satisfied as much as possible.

Step5. Check if the system designing has been satisfactory. If it is, stop tuning, else change Q_N,Q,N and go to step2.

Take single input system with only symmetric input saturation constraint as an example. Fig. 3 is the curve of $h(x^L)$. Now (19) becomes $B^{\mathrm{T}}P_1B/\lambda \leqslant (2b_1-1)/(b_1-1)^2$ for which the necessary condition is $b_1>1/2$. Since $x_{\max}^L=\max_{k\geqslant 0}|x_k^L|$, if $b_1\leqslant 1/2$, we can increase λ to decrease x_k^L and consequently decrease $x_{\max}^L$ to make $b_1>1/2$.

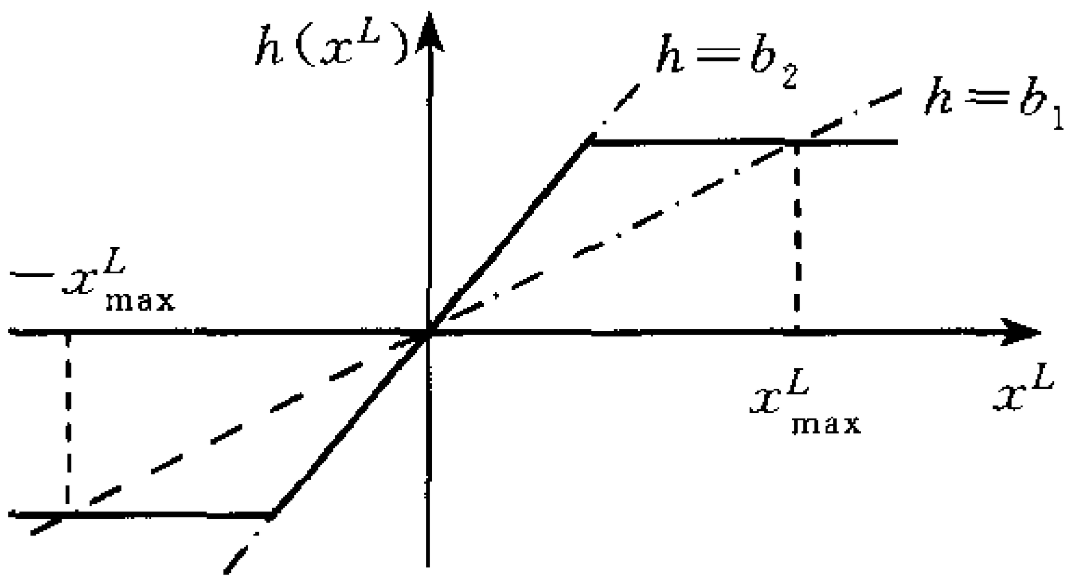

Fig. 3 The sketch map shows the saturation's restriction on b_1

Regarding the above parameter tuning guideline, we can give the following conclusion to show that TSMPC is tunable whenever A is stable.

Theorem 2. For system (1) with stable A, assume that $\delta(b,b_1)>0$ if there isn't input saturation constraint. Then, by tuning $\{Q_N,Q,\lambda,N\}$, all the conditions in Corollary 1 (or Corollary 2) can be satisfied.

Proof. Take Corollary 2 as an example. Corollary 1 will be analogous. At first, if saturation constraint is not considered, then determining $\delta(b,b_1)$ is independent of control parameters. When there is saturation constraint, consider the following two cases:

Case A: As $\lambda=\lambda_0, b_1>0.5$. Always take $Q_N=Q+A^{\mathrm{T}}Q_NA-A^{\mathrm{T}}Q_NB(\lambda I+B^{\mathrm{T}}Q_NB)^{-1}B^{\mathrm{T}}Q_NA$. This is equivalent to infinite horizon control[5], so $P_0-P_1=0$. Further take $Q>$

0, then $Q > P_0 - P_1$. Moreover, since A is stable, P_1 will be bounded for any $\lambda \leqslant \infty$. Hence, there exists sufficient large λ_1 such that whenever $\lambda \geqslant \lambda_1 \geqslant \lambda_0$, $\lambda(2b_1 - 1) \geqslant (b-1)^2 \sigma_{\max}(B^T P_1 B)$ and all the conditions in Corollary 2 are satisfied.

Case B: As $\lambda = \lambda_0$, $b_1 \leqslant 0.5$. Since $\delta(b, b_1) > 0$ if there isn't saturation constraint, $b_1 \leqslant 0.5$ is only due to the heavy constraints on control actions by saturation. According to equation (5) and the reason stated in Case A we can conclude: for any arbitrary large $\| z_k \|$, there is a sufficiently large λ_2, such that whenever $\lambda \geqslant \lambda_2 \geqslant \lambda_0$, $\hat{u}_k$ doesn't violate saturation constraint, that is, take $\lambda \geqslant \lambda_2$ then $b_1 > 0.5$.

In a word, if $\lambda = \lambda_0$ doesn't satisfy the conditions in Corollary 2, then choosing $\lambda \geqslant \max\{\lambda_1, \lambda_2\}$ and suitable $\{Q_N, Q, N\}$ may satisfy it. Therefore, the theorem holds. □

The concrete design procedure is referred to the following simulation example.

5 Simulation example

The system adopted is open-loop unstable, where

$$A = \begin{bmatrix} 1.2 & 0.34 & 0.12 & 1 \\ 1 & 2 & 1 & 0.34 \\ 0.23 & 0.23 & 2 & 0.34 \\ 0.3 & 0.3 & 0.4 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 0.23 & 0.89 \\ 1 & 1 \\ 0.3 & 1.2 \end{bmatrix} \text{ and } C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

It is easy to know that the linear system is controllable and observable. The Hammerstein nonlinearity is described by

$$x_1 = \phi_1(u_1) = \begin{cases} 0.0448\theta^5 - 0.2512\theta^3 + 1.2802\theta, & |\theta| \leqslant 2 \\ 0.9922\theta, & |\theta| > 2 \end{cases}$$

and $f_2(\theta) = \theta$. The saturation constraints are $-1 \leqslant u_1 \leqslant 1$ and $-1 \leqslant u_2 \leqslant 1$. $sat_i\{\theta\} = \text{sign}\{\theta\} \cdot \min\{1, |\theta|\}, i = 1, 2$.

Adopt TSMPC proposed in this paper. The initial state is $z_0 = [0.55, -0.6, 0.55, -0.6]$. Tune the system by applying the guideline Step1~Step5 given above. In the simulation, the linear part always satisfies condition i) in Theorem 1. And since u_2 has never saturated in the simulation, in the following we concentrate on the computation of u_1.

Case a. Hammerstein nonlinearity is not considered in computation of u_1, that is, $g_1 = sat_1$. The parameters are chosen as $N=4$, $\lambda = 0.005$, $Q = C^T C$, $Q_N = 0.1 I_4 + Q + A^T Q_N A - A^T Q_N B(\lambda I_2 + B^T Q_N B)^{-1} B^T Q_N A$ where $I_2(I_4)$ is second-order (fourth-order) identity matrix. This choice of Q_N is equivalent to quasi-infinite horizon control[6].

Case b. Consider Hammerstein nonlinearity, that is, the formula used to calculate u_1 is

$$u_1 = g_1(x_1^L) = \begin{cases} sat_1\{-0.0488(x_1^L)^5 + 0.2674(x_1^L)^3 + 0.7075 x_1^L\}, & |x_1^L| \leqslant 2 \\ sat_1\{0.9963 x_1^L\}, & |x_1^L| > 2 \end{cases}$$

Other parameters are chosen the same as Case a.

Case c. Same as Case b except that $\lambda = 20$.

The simulation results are shown in Fig. 4, where (a), (b), (c) correspond to the above three cases, and 1, 2 to system output and input respectively. Solid and dotted lines mean two output (input) variables. Moreover, to illustrate whether all the conditions in Theorem 1 are satisfied, Fig. 4(a_3), (b_3) and (c_3) protract $c_k = -\lambda(\boldsymbol{x}_k^L)^T(2\boldsymbol{x}_k - \boldsymbol{x}_k^L) + (\boldsymbol{x}_k - \boldsymbol{x}_k^L)^T B^T P_1 B(\boldsymbol{x}_k - \boldsymbol{x}_k^L)$, named as stability condition testing curves. That is, when $c_k \leqslant 0$, the condition ii) in Theorem 1 is satisfied at time k.

For Case a, according to the stability condition testing curve we know that the condition ii) in Theorem 1 is not satisfied. The simulation results Fig. 4(a_1) and (a_2) also show that the system is unstable. For Case b, h_1 does not satisfy the condition ii) in Theorem 1. The system is stable but is not of good quality. And for Case c, the conditions in Theorem 1 are always satisfied. The system is stable and the control quality is better than in (b). The simulation results above illustrate the conclusions in Theorem 1.

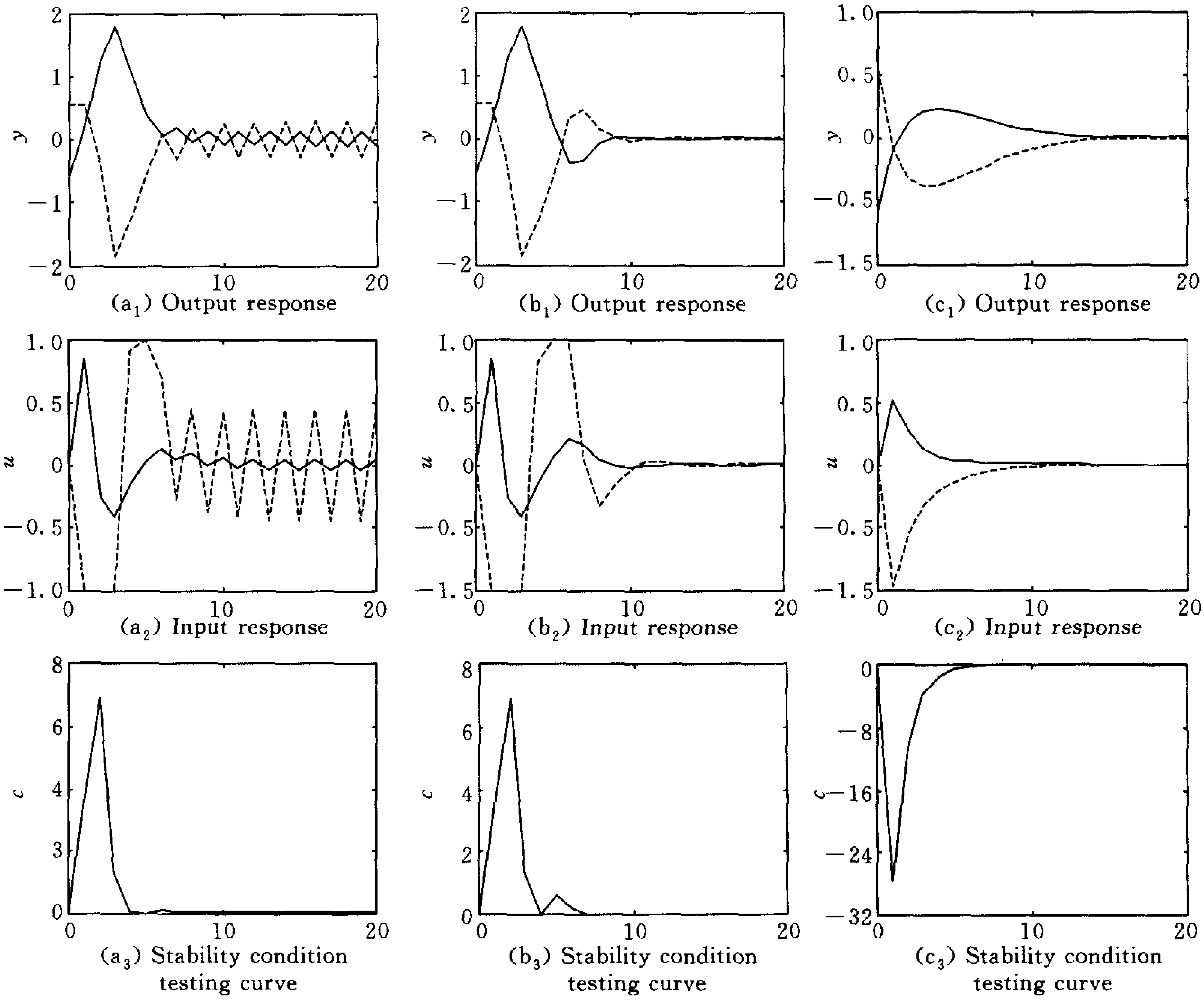

Fig. 4 Simulation results, where the abscissa is the sample time

$(a_1)\sim(a_3)$ $\lambda=0.005$, not considering Hammerstein nonlinearity; $(b_1)\sim(b_3)$ $\lambda=0.005$, considering Hammerstein nonlinearity; $(c_1)\sim(c_3)$ $\lambda=20$, considering Hammerstein nonlinearity

6 Conclusion

This paper studies a two-step predictive controller for systems with input nonlinearities, including Hammerstein nonlinearity, saturation constraint and static uncertainty. Some stability conclusions are obtained. Most of the existing stable predictive controllers dealing with real constraints add extra artificial constraints and discuss the feasibility problem after having guaranteed the stability[7,8]. Deducing the stability property of TSMPC for systems with constraints and/or input nonlinearities, on the other hand, starts from the feasible linear control law, and its main problem is to investigate the domain of attraction of this kind of systems. The fact that the latter does not add artificial constraints is a main characteristic of the studies in this paper.

Deducing the domain of attraction based-on (8) needs further investigation. Moreover, TSMPC with state observer (i. e., output feedback TSMPC), robustness of TSMPC for the linear part with uncertainties, *etc.*, all deserve further investigation. The results in this paper can serve as a basis for these future studies.

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具有输入非线性的离散时间系统预测控制的稳定性分析

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摘　要　对存在输入非线性的系统，采用两步法预测控制策略. 对线性部分采用 Riccati 迭代矩阵满足一定条件的控制律以得到 Lyapunov 函数，进而研究了存在非线性反算误差（由非线性方程求解误差和解饱和算法形成）时两步法使系统保持稳定的条件，给出了实际系统参数调整的指导方法. 通过仿真说明了理论分析结果的有效性.

关键词　输入非线性，两步法，预测控制，Riccati 迭代，稳定性

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