

The Design and Stability Analysis of an Adaptive System Based on Linear T-S Fuzzy System¹⁾

LI Li LI Yong-Ming

(Department of Automatic Control, Northwestern Polytechnical University, Xi'an 710072)

(E-mail: Maggie_ll@163.com)

Abstract An adaptive fuzzy controller of a class of affine nonlinear systems is presented. The approximators using linear T-S fuzzy logic system (FLS) are adopted to estimate the unknown functions and the tunable parameters are the coefficients of the consequent rules. Because of better estimating performance of linear T-S FLS, we can achieve a smaller tracking error than using Mamdani FLS. The closed-loop system can get global stability by using Lyapunov synthesis approach in the sense that all the signals involved are uniformly bounded. Simulation of inverted pendulum tracking is carried out to verify the design.

Key words Adaptive fuzzy system, Takagi-Sugeno fuzzy system, affine nonlinear system control

1 Introduction

The application of fuzzy theory to control problems has been the focus of numerous studies^[1]. The motivation is that the fuzzy set theory provides an alternative to the traditional modeling and design of control systems, where system knowledge and dynamic models are uncertain and time varying in the traditional sense. Recently, some researches have been directed at the use of the Lyapunov synthesis approach to construct stable adaptive fuzzy controllers^[2~5]. Most of the parameterized fuzzy approximators are expressed as a series of radial basis function (RBF) expansion^[2~6], and most of them are based on Mamdani FLS^[2~4].

However, [7] has proved that T-S fuzzy systems with linear rule consequences are universal approximators and are more capable of approximating functions than Mamdani FLS. So in this paper we will present a class of fuzzy approximators using T-S FLS of linear rule consequences. The tunable parameters are the coefficients of consequent rules. In order to achieve global stability, Lyapunov synthesis approach is used. Some tactics are adopted to guarantee $\hat{g} > 0$ in part 4 and part 5.

2 Control objective

Consider the n th-order nonlinear systems of the form:

$$\dot{x}^{(n)} = f(x, \dot{x}, \dots, x^{(n-1)}) + g(x, \dot{x}, \dots, x^{(n-1)})u, \quad y = x \quad (1)$$

where f and g are unknown continuous functions. $\mathbf{x} = (x_1, \dots, x_n)^T = (x, \dots, x^{(n-1)})^T \in \mathbb{R}^n$ is the state vector of the system, which is assumed to be available for measurement. For (1) to be controllable, we require that $g(\mathbf{x}) \neq 0$ for \mathbf{x} in certain controllability region $U_c \in \mathbb{R}^n$. Since $g(\mathbf{x})$ is continuous, we can assume that $g(\mathbf{x}) > 0$ for $\mathbf{x} \in U_c$. The control objective is:

1) The closed-loop system must be globally stable and robust in the sense that all variables, $\mathbf{x}(t)$, $\boldsymbol{\theta}(t)$, and $\mathbf{u}(\mathbf{x}|\boldsymbol{\theta})$ are bounded, i. e., $\|\mathbf{x}(t)\| \leq M_x < \infty$, $\|\boldsymbol{\theta}(t)\| \leq M_\theta < \infty$, $\|\mathbf{u}(\mathbf{x}|\boldsymbol{\theta})\| \leq M_u < \infty$ for all $t \geq 0$, where M_x, M_θ, M_u are design parameters.

2) The tracking error $\mathbf{e} = \mathbf{y}_m - \mathbf{y}$ should be as small as possible under the constraints in 1).

1) Supported by National Natural Science Foundation of P. R. China(60174016), and the Key Project of Fundamental Research(2002CB312200)

Received January 4, 2002; in revised form December 20, 2002

收稿日期 2002-01-04; 收修改稿日期 2002-12-20

3 Description of linear T-S fuzzy systems

Because $f(x)$ and $g(x)$ are unknown, T-S fuzzy systems are used to approximate them.

Assume that the fuzzy rule bases of $\hat{f}(x)$ consist of a collection of fuzzy IF—THEN rules, R_f^l : IF (x_1 is A_1^l , and, \dots , and x_n is A_n^l) then $\hat{f}(x) = a_0^l + \dots + a_n^l x_n$ ($l=1, \dots, m$).

Assumption 1. Suppose the fuzzy sets satisfy when $A_i^l = 1, \sum_{\substack{j=1 \\ j \neq i}}^m A_j^l \ll 1$, which means:

if we choose triangle membership function, then the fuzzy sets are normal and complete; and

if we choose Gaussian membership function, it satisfy that $x_i^l - x_j^l \gg \sigma$, where x_i^l, x_j^l are the centers and σ is the variance of the functions.

Consider a subset of the fuzzy system with product inference, Gaussian membership and the center-average defuzzifier. Such a fuzzy system can be expressed as the following:

$$\hat{f}(x | \theta_f) = \frac{\sum_{l=1}^m (a_0^l + \sum_{i=1}^n a_i^l x_i) \prod_{i=1}^n \exp\left(-\left(\frac{x_i - \bar{x}_i^l}{\sigma_i^l}\right)^2\right)}{\sum_{l=1}^m \prod_{i=1}^n \exp\left(-\left(\frac{x_i - \bar{x}_i^l}{\sigma_i^l}\right)^2\right)} = \sum_{l=1}^m \xi^l(x) \theta_f^l \mathbf{X} = \xi(x) \theta_f \mathbf{X}$$

with $\theta_f^l = (a_0^l, \dots, a_n^l) \in R^{1 \times (n+1)}$, $\mathbf{X} = [1, x_1, \dots, x_n]^T$

$$\xi^l(x) = \frac{\prod_{i=1}^n \exp\left(-\left(\frac{x_i - \bar{x}_i^l}{\sigma_i^l}\right)^2\right)}{\sum_{l=1}^m \prod_{i=1}^n \exp\left(-\left(\frac{x_i - \bar{x}_i^l}{\sigma_i^l}\right)^2\right)}$$

$$\xi(x) = (\xi^1(x), \dots, \xi^m(x)) \in R^{1 \times m} \quad \theta_f = [\theta_f^{1T}, \dots, \theta_f^{mT}]^T \in R^{m \times (n+1)} \quad (2)$$

$\hat{g}(x)$ can be expressed similarly.

4 Adaptive controller

Assumption 2. We can determine functions $f^u(x)$ and $g_l(x)$ such that $|f(x)| < f^u(x)$ and $g_l(x) < g(x)$ for $x \in U_c$, where $f^u(x) < \infty$, and $g_l(x) > 0$.

Let $e = (e, \dot{e}, \dots, e^{(n-1)})$ and $k = (k_n, \dots, k_1)^T \in R^n$ be such that all roots of the polynomial $h(s) = s^n + k_1 s^{n-1} + \dots + k_n$ are in the open left-half plane. Since we can get the error equation

$$\dot{e} = \Lambda_c e + b_c [(\hat{f} - f) + (\hat{g} - g)u_c]$$

with

$$\Lambda_c = \begin{bmatrix} 0 & 1 & \dots & \dots & 0 \\ 0 & 0 & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ \vdots & \vdots & & & 1 \\ -k_n & \dots & \dots & \dots & -k_1 \end{bmatrix}_{n \times n} \quad b_c = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}_{n \times 1}$$

Since $|SI - \Lambda_c| = s^n + k_1 s^{n-1} + \dots + k_n$ is stable, there must exist a unique positive definite symmetric $n \times n$ matrix P which satisfies the Lyapunov equation $\Lambda_c^T P + P \Lambda_c = -Q$, where Q is an arbitrary $n \times n$ positive definite matrix.

We choose the control law $u = u_c + u_s$, where $u_c = \frac{1}{\hat{g}(x | \theta_g)} [-\hat{f}(x | \theta_f) + y_m^{(n)} + k^T e]$,

$$u_s = \left(\frac{V_e}{\bar{V}}\right)^p \text{sgn}(e^T P b_c) \left[|u_c| + \frac{1}{g_l} (f^u(x) + |y_m^{(n)}| + |k^T e|) \right] \text{ with } p = 1, 2, 3, \text{ or } 4 \quad (3)$$

The adaptive law of the tunable parameters is:

$$(\dot{\theta}_f^l)^T = -\Gamma_f^l e^T P b_c \xi^l(x) \mathbf{X}, \quad (\dot{\theta}_g^l)^T = -\Gamma_g^l e^T P b_c \xi^l(x) \mathbf{X} u_c \quad (4)$$

The analysis of stability will be developed in the next part.

5 Stability Analysis

Define $\theta_f^* = \arg \min_{\theta_f \in \Omega_f} [\sup_{x \in u_c} |\hat{f}(x | \theta_f) - f(x)|]$

$\theta_g^* = \arg \min_{\theta_g \in \Omega_g} [\sup_{x \in u_c} |\hat{g}(x | \theta_g) - g(x)|]$

where Ω_f and Ω_g are constraint sets for θ_f and θ_g , respectively, specified by the designer.

We need $\Omega_f = \{\theta_f^l : \|\theta_f^l\| \leq M_f\}$ and $\Omega_g = \{\theta_g^l : \|\theta_g^l\| \leq M_g, \xi(x)\theta_g^l X > \epsilon\}$, where M_f, M_g, ϵ are positive constraints specified by the designer.

We can use the following tactics to guarantee $\hat{g} > \epsilon$.

Assume for $x_i \in \Omega_i$ there exists a membership function including point $(0, \xi^r(0))$ and the set is $\Omega_{x_i}^r$ accordingly. Suppose the width of this set is c .

Let $b_0^l \geq 2\epsilon$. When $x_i \in \Omega_{x_i}^r$, let $|b_i^r| \leq \frac{\epsilon}{nc}$.

When $x_i \in \Omega_{x_i}^l$ and in this set $x_i < 0$, let $b_i^l < -\epsilon$.

When $x_i \in \Omega_{x_i}^l$ and in this set $x_i > 0$, let $b_i^l > \epsilon$.

Suppose h is the number of $x_i \in \Omega_{x_i}^r$; then we always have

$$b_0^l + b_1^l x_1 + \dots + b_n^l x_n \geq b_0^l + \underbrace{b_i^r x_i + \dots + b_j^r x_j}_h \geq |b_0^l| - |b_i^r x_i| - \dots - |b_j^r x_j| \geq$$

$$|b_0^l| - c[|b_i^r| + \dots + |b_j^r|] \geq 2\epsilon - c \cdot h \cdot \frac{\epsilon}{nc} = \left(2 - \frac{h}{n}\right)\epsilon \geq \epsilon$$

We can get $\hat{g} = \sum_{l=1}^m \xi^l(x)(b_0^l + b_1^l x_1 + \dots + b_n^l x_n) > \sum_{l=1}^m \xi^l(x)\epsilon = \epsilon$, so $\hat{g} > \epsilon$.

Define the minimum approximation error as follows:

$$\omega = \hat{f}(x | \theta_f^*) - f(x) + (\hat{g}(x | \theta_g^*) - g(x))u_c \tag{5}$$

Suppose $\phi_f^l = \theta_f^l - \theta_f^{l*} \in R^{1 \times (n+1)}$, $\phi_g^l = \theta_g^l - \theta_g^{l*} \in R^{1 \times (n+1)}$

Let

$$V = \frac{1}{2}e^T P e + \sum_{l=1}^m \phi_f^l \times \frac{1}{2}(\Gamma_f^l)^{-1}(\phi_f^l)^T + \sum_{l=1}^m \phi_g^l \times \frac{1}{2}(\Gamma_g^l)^{-1}(\phi_g^l)^T$$

$$\dot{V} = -\frac{1}{2}e^T Q e + e^T P b_c \omega - e^T P b_c g(x)u_s + e^T P b_c \left[\sum_{l=1}^m \xi^l(x)\phi_f^l X + \left(\sum_{l=1}^m \xi^l(x)\phi_g^l X \right) u_c \right] +$$

$$\sum_{l=1}^m \phi_f^l (\Gamma_f^l)^{-1}(\dot{\phi}_f^l)^T + \sum_{l=1}^m \phi_g^l (\Gamma_g^l)^{-1}(\dot{\phi}_g^l)^T = -\frac{1}{2}e^T Q e - g(x)e^T P b_c u_s + e^T P b_c \omega.$$

Applying u_s as (3), we can prove just as in [4]

$$\dot{V} \leq -\frac{1}{2}e^T Q e + e^T P b_c \omega.$$

Remark. If $\omega = 0$, i. e., the searching spaces for \hat{f} and \hat{g} are so big that f and g are included in them, then we have $\dot{V} \leq 0$. Because the fuzzy systems are universal approximators, we can hope that ω should be small, if not equal to zero, provided that we use sufficiently complex \hat{f} and \hat{g} .

To guarantee $\theta_f^l \in \Omega_f$ and $\theta_g^l \in \Omega_g$, we use parameter projection algorithm as stated in [8].

Besides that, in order to guarantee $\hat{g} > \epsilon$, we should present such tactics:

When $x_i \in \Omega_{x_i}^l$ and in this set $x_i < 0$, if $b_i^l = -\epsilon$,

$$\dot{b}_i^l = -\tau_{g_i}^l e^T P b_c \xi^l(x)x_i u_c \quad \text{when } e^T P b_c \xi^l(x)u_c < 0$$

$$\dot{b}_i^l = 0 \quad \text{when } e^T P b_c \xi^l(x)u_c \geq 0$$

When $x_i \in \Omega_{x_i}^l$ and in this set $x_i > 0$, if $b_i^l = \epsilon$,

$$\begin{aligned} \dot{b}_i^l &= -\tau_{gi}^l e^T P b_c \xi^l(x) x_i u_c & \text{when } e^T P b_c \xi^l(x) u_c < 0 \\ \dot{b}_i^l &= 0 & \text{when } e^T P b_c \xi^l(x) u_c \geq 0. \end{aligned}$$

When $x_i \in \Omega_{x_i}^r$ and if $|b_i^r| = \frac{\varepsilon}{nc}$

$$\begin{aligned} \dot{b}_i^r &= -\tau_{gi}^r e^T P b_c \xi^r(x) x_i u_c & \text{when } e^T P b_c \xi^r(x) u_c > 0 \\ \dot{b}_i^r &= 0 & \text{when } e^T P b_c \xi^r(x) u_c \leq 0 \end{aligned}$$

when $b_0^r = 2\varepsilon$,

$$\begin{aligned} \dot{b}_0^r &= -\tau_{g0}^r e^T P b_c \xi^r(x) u_c & \text{when } e^T P b_c \xi^r(x) u_c < 0 \\ \dot{b}_0^r &= 0 & \text{when } e^T P b_c \xi^r(x) u_c \geq 0 \end{aligned} \quad (6)$$

Theorem:

- 1) $\|\theta_f^l\| < M_f, \|\theta_g^l\| < M_g$
- 2) $|x(t)| \leq |y_m| + \left(\frac{2\bar{V}}{\lambda_{\min}}\right)^{1/2}$, where λ_{\min} is the minimum eigenvalue of P
- 3) $|u| \leq \frac{1}{g_l} (f^u(x) + |y_m^{(n)}| + |k^T e|) + \frac{2}{\varepsilon} \left[m \left(|y_m| + \left(\frac{2\bar{V}}{\lambda_{\min}}\right)^{1/2} \right) M_f + |y_m| + \|k\| \left(\frac{2\bar{V}}{\lambda_{\min}}\right)^{1/2} \right]$
- 4) $\int_0^t |e(\tau)|^2 d\tau \leq a + b \int_0^t |\omega(\tau)|^2 d\tau$
- 5) If $\omega \in L_2$, that is, $\int_0^t |\omega(\tau)|^2 d\tau < \infty$, then $\lim_{t \rightarrow \infty} |e(t)| = 0$

The proof of this theorem is similar to that of [2] and [4], here we omit it.

Remark. In [6], it is assumed that we can find such constraints of the tunable parameters of \hat{g} that in these constraints \hat{g} will always be positive. Projection algorithm is applied to guarantee that the parameters are within the constraints. However, since the parameters are relative to the states of systems, it is very difficult to determine the bounds. In this paper, we adopt the above tactics in the adapting process to guarantee $\hat{g} > 0$ and it can be easily carried out in practice.

7 Simulation

In this section, we use our adaptive fuzzy controllers to control the inverted pendulum to track a sine wave trajectory. As in [9], the dynamic equations of the inverted pendulum system are:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \frac{g \sin x_1 - \frac{m l x_2^2 \cos x_1 \sin x_1}{m_c + m}}{l \left(\frac{3}{4} - \frac{m \cos^2 x_1}{m_c + m} \right)} + \frac{\frac{\cos x_1}{m_c + m}}{l \left(\frac{3}{4} - \frac{m \cos^2 x_1}{m_c + m} \right)} u \end{aligned} \quad (7)$$

where $g = 9.8 \text{ m/s}^2, m_c = 1 \text{ kg}, m = 0.1 \text{ kg}$ and $l = 0.5 \text{ m}$. We choose the reference signal $y_m(t) = \frac{\pi}{30} \sin(t)$ in the following simulation.

$$|f(x_1, x_2)| \leq 15.78 + 0.0366 x_2^2 = f^u(x_1, x_2).$$

If we require $|x| \leq \frac{\pi}{6}, |g(x_1, x_2)| \geq 1.12 = g_l(x_1, x_2)$

Suppose $\bar{V} = 0.267, M_f = 16, M_g = 1.6$ and $\varepsilon = 0.7$. The absolute error of it is:

$$IAE1 = \int_{t_0}^{t_f} |x_1 - x_d| dt = 0.0603 \text{ and } IAE2 = \int_{t_0}^{t_f} |x_2 - \dot{x}_d| dt = 0.0858$$

However, if we use the controllers with Mamdani fuzzy system as in [2] and the same initial conditions as the above controllers, then the absolute error will be

$$IAE1 = \int_{t_0}^{t_f} |x_1 - x_d| dt = 0.0831 \text{ and } IAE2 = \int_{t_0}^{t_f} |x_2 - \dot{x}_d| dt = 0.1108$$

From Fig. 1, Fig. 2 and the absolute error, we can see that the controllers using linear T-S fuzzy system have better approximating property than the ones using Mamdani fuzzy system.

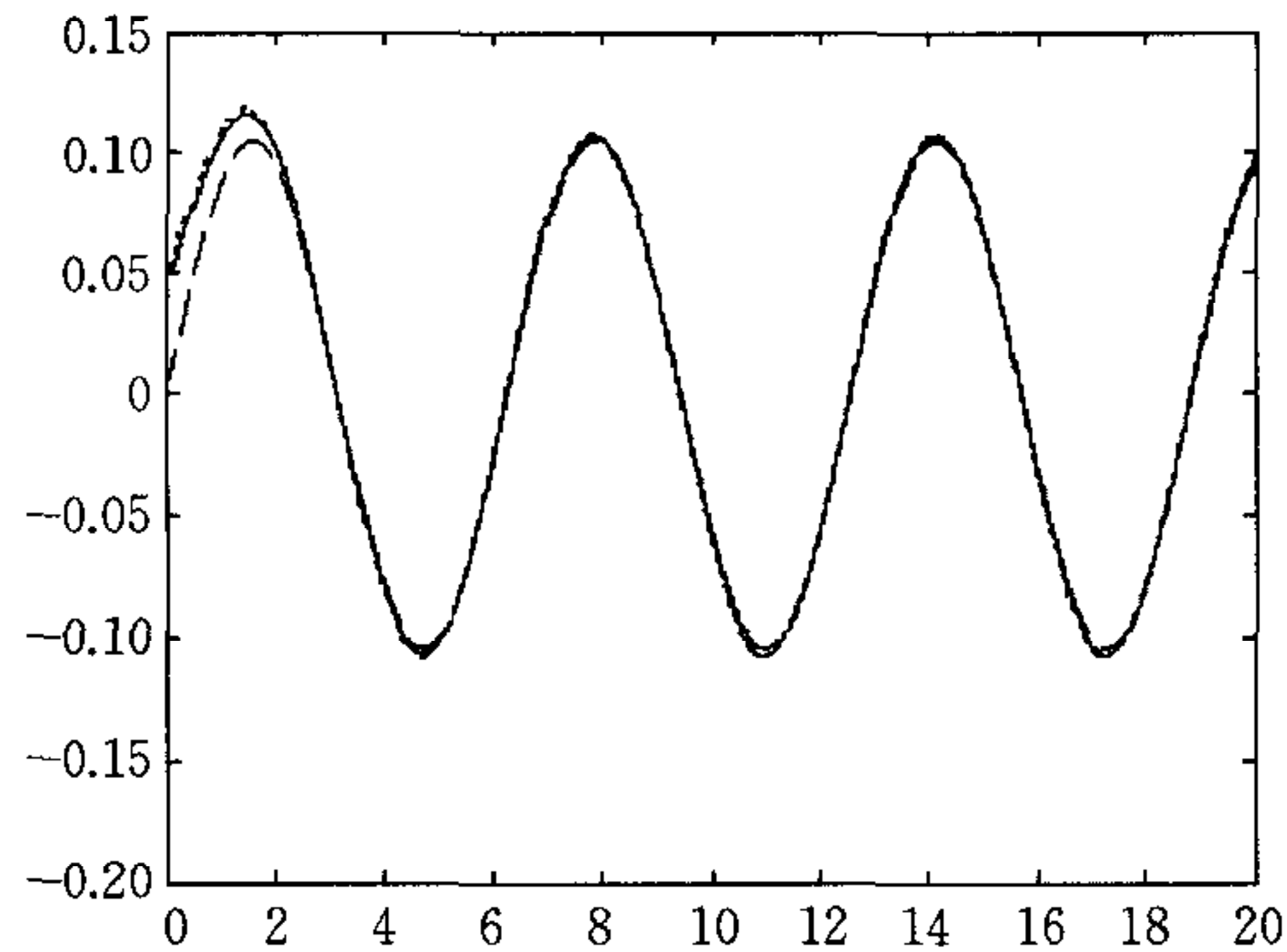


Fig. 1 the state x_1 (solid line) in this paper , the state x_1 (dotted line) in [2] and its desired value $y_m(t) = \frac{\pi}{30} \sin(t)$ (dashed line) for initial condition $x(0) = (\frac{\pi}{60}, 0)$

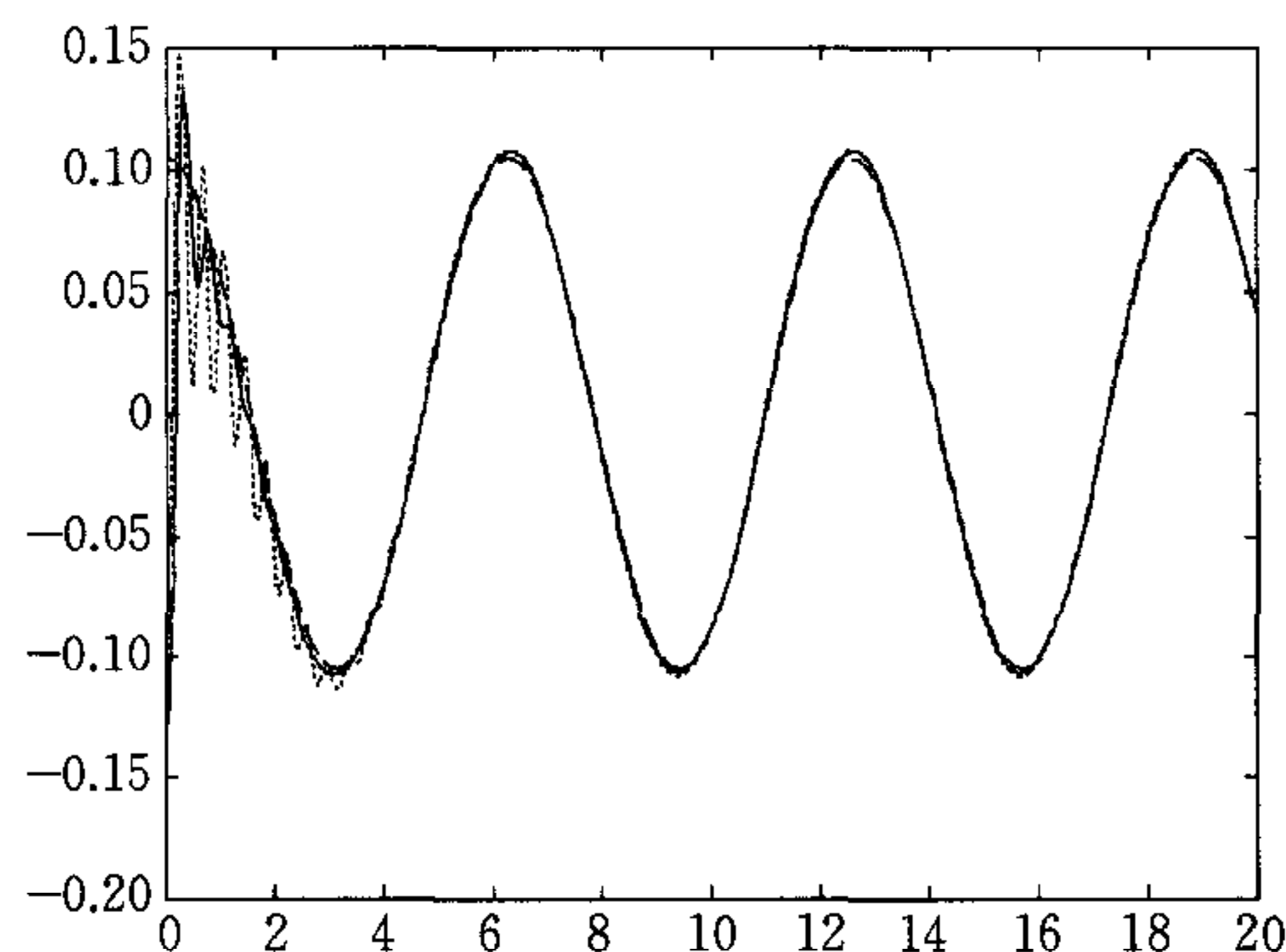


Fig. 2 the state x_2 (solid line) in this paper, the state x_2 (dotted line) in [2] and its desired value $y_m(t) = \frac{\pi}{30} \cos(t)$ (dashed line) for initial condition $x(0) = (\frac{\pi}{60}, 0)$

8 Conclusion

In this paper, we have developed an adaptive fuzzy controller to provide asymptotic tracking to a reference signal for a class of affine nonlinear systems. We can achieve a better tracking precision since the approximators we use to approximate the unknown functions of the system are constructed on the basis of linear T-S FLS. The global stability of the resulting closed-loop systems is guaranteed in the sense that all signals involved are uniformly bounded. Simulation of the inverted pendulum demonstrates the property of this controller.

References

- 1 Lee C C. Fuzzy logic in control systems: Fuzzy logic controller, Part I and Part II. *IEEE Transactions on Systems, Man, Cybernetics*, 1990, **20**(2): 404~435
- 2 Wang L X. Stable adaptive fuzzy controller with application to inverted pendulum tracking. *IEEE Transactions on Systems, Man, Cybernetics*, 1996, **26**(5): 677~691
- 3 Han Hu-Gang, Su Chun-Yi, Stepanenko Yury. Adaptive control of a class of nonlinear systems with nonlinearly parameterized fuzzy approximators. *IEEE Transactions on Fuzzy Systems*, 2001, **9**(2): 315~323
- 4 Zhang Wei-Guo, Yang Xiang-Zhong, Duan Fu-Hai. On improving performance of controlled system with continuous fuzzy monitoring. *Journal of Northwestern Polytechnical University*, 1999, **17**(3): 371~375 (in Chinese)
- 5 Tsay Dong-Ling, Chung Hung-Yuan, Lee Ching-Jung. The adaptive control of nonlinear systems using the Sugeno-Type of fuzzy logic. *IEEE Transactions on Fuzzy Systems*, 1999, **7**(2): 225~229
- 6 Spooner J T, Passino K M. Stable adaptive control using fuzzy systems and neural networks. *IEEE Transactions on Fuzzy Systems*, 1996(4): 339~359
- 7 Ying H. General SISO Takagi-Sugeno fuzzy systems with linear rule consequent are universal approximators. *IEEE Transactions on Fuzzy Systems*, 1998, **6**(6): 582~587
- 8 Cuenberger D G. *Linear and Nonlinear Programming*. Reading, MA: Addison-Wesley, 1984
- 9 Sloting J E, Li W. *Applied Nonlinear Control*. Englewood Cliffs, NJ: Prentice-Hall, 1991

LI Li Received her bachelor degree in 2000. Now she is a master student at the Northwest Polytechnical University. Her research interests include fuzzy adaptive control.

LI Yong-Ming Received his Ph. D. degree in 1996. Completed his postdoctoral work at the Northwest Polytechnical University during 1997~1999. His research interests include topology, fuzzy mathematics, and control.

基于线性 T-S 模糊系统的自适应系统的设计和稳定性分析

李 莉 李永明

(西北工业大学自动控制系 西安 710072)

(E-mail: Maggie-ll@163.com)

摘 要 提出了一种新型的模糊自适应控制器. 其中系统的未知函数由线性 T-S 模糊系统估计. 模糊规则的后件是线性可调的. 因线性 T-S 模糊系统较之 Mamdani 模糊系统具有更好的估计性能, 我们可以获得更小的跟踪误差. 由于采用李亚普诺夫合成方法来设计控制器, 整个闭环系统全局稳定, 所有的信号有界. 对倒立摆的仿真验证了这一点.

关键词 模糊自适应, Takagi-Sugeno 模糊系统, 仿射非线性系统控制

中图分类号 TP13