Average Dwell-time Conditions for Consensus of Discrete-time Linear Multi-agent Systems with Switching Topologies and Time-varying Delays

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Abstract This paper investigates the consensus problem of discrete-time linear multi-agent systems (DLMASs) with directed switching information topologies and time-varying delays. First, we transform the consensus problem to an asymptotic stability problem of a corresponding time-delayed switched linear system (TDSLS) via a proper linear transformation. Then, by using a constructed Lyapunov functional and the average dwell-time scheme, we establish a novel delay-dependent sufficient condition for the solvability of the consensus problem in terms of linear matrix inequalities (LMIs) for two cases, respectively: 1) all of the given information topologies are consensusable; 2) some of the given information topologies are consensusable. Finally, numerical examples are given to show the validity of the established results.

Key words Multi-agent systems (MASs), consensus, directed switching information topologies, time-varying delay, average dwell-time, linear matrix inequalities (LMIs)


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The problem of distributed coordinated control of multi-agent systems (MASs) has been widely studied by many researchers due to its broad applications in many areas\textsuperscript{[13]–[18]}. A critical problem is to design a control protocol for each agent, using only local information from its neighbors, such that all agents achieve an agreement on certain quantities of interest. This problem is usually called the consensus problem\textsuperscript{[9–21]}. Switching information topologies are ubiquitous in many practical applications, e.g., link failures or creations in networks with mobile nodes. Moreover, time delay is a ubiquitous phenomenon in networks due to the limited speed of signals traveling through links. In this paper, we investigate consensus problem of discrete-time linear multi-agent systems (DLMASs) with directed switching information topologies and time-varying delay, where the lower and upper bounds of the delay are known. Here we give an overview of the consensus problem of DLMASs.

Based on graph theory, \textsuperscript{[22]} concluded that consensus of first-order DLMASs with switching information topologies can be achieved asymptotically if the union of the information topologies is connected frequently enough as the system evolves. Reference \textsuperscript{[12]} extended the results in \textsuperscript{[22]} to the case of directed switching information topologies and showed that information consensus can be achieved asymptotically if the union of the directed information topologies has a spanning tree frequently enough as the system evolves. Reference \textsuperscript{[13]} proved that the average-consensus of DLMASs with directed switching information topologies can be achieved if at each instant the switching function topology is balanced and the union of the information topologies is connected. The above researches supposed that there are no communication time-delays. The consensus problem of first-order DLMASs with both time-varying delays and switching information topologies was addressed in \textsuperscript{[14–15]}. The extension to the second-order DLMASs was done in \textsuperscript{[16–17]}. Both the leaderless consensus problem and the leader-following consensus problem for high-order DLMASs under switching information topologies were studied in \textsuperscript{[18]}. In the above literatures, the condition for consensus of DLMASs with switching information topologies is expressed in terms of the connectivity of the union of the information topologies. However, there are some problems demanded to discuss and resolve, such as how to design the switching order of the given information topologies and determine the dwell-time on each of the information topologies. In many investigations\textsuperscript{[16–17,19–21,23–24]}, the consensus problem was transformed to the asymptotic stability problem by using a proper transformation. Reference \textsuperscript{[23]} transformed the exponential synchronization problem of the continuous-time nonlinear dynamic network into the problem of exponential stability of the error dynamical systems by introducing error vector, and studied the local and global exponential synchronization of the dynamical network with undirected switching information topologies and time-varying coupling delays by using the average dwell-time scheme. However, the transformation in \textsuperscript{[23]} is not suitable for the case of directed information topology. Thus \textsuperscript{[24]} presented a proper linear transformation and transformed the consensus problem of linear MASs to the problem of exponential stability, and gave average dwell-time conditions of the consensus problem for continuous-time linear MASs under directed switching information topologies but without time-delay.

Motivated by \textsuperscript{[23–24]}, we transform the consensus problem into the stability problem of a corresponding time-delayed switched linear system (TDSLS) by using a linear transformation similar to that proposed in \textsuperscript{[11,24]}. On the other hand, the average dwell-time scheme has been shown to be an effective tool in the study of stability of switched systems\textsuperscript{[25–29]}. Then, we apply the stability the-
ory of TDSLS to deduce the switching rules of the information topologies for achieving global state consensus. By using a constructed Lyapunov functional and the average dwell-time scheme, we establish a novel delay-dependent sufficient condition in terms of linear matrix inequalities (LMIs) for the solvability of the consensus problem of DLMASs with time-varying delays under directed switching information topologies for two cases, respectively: 1) all of the given information topologies are consensusable (see Definition 4 in Section 1); 2) some of the given information topologies are consensusable. Compared with the existing works in [23–24], we obtain the average dwell-time condition for the consensus problem of general high-order DLMASs with directed switching information topologies and time-varying delay.

The rest of this paper is organized as follows. The consensus problem of DLMASs with directed switching information topologies and time-varying delays and the linear transformation are presented in Section 1. In Section 2, a sufficient condition for the consensus problem is established in terms of LMIs under two cases. The effectiveness of the proposed approach is illustrated with numerical examples in Section 3. Section 4 concludes this paper.

1 Problem description and linear transformation

Before stating the consensus problem we give some basic notations. Superscript “T” denotes the transpose of a matrix. $\mathbb{R}^n$ and $\mathbb{R}^{m \times n}$ stand for the set of real vectors of dimension $n$ and the real matrix of size $n \times m$, respectively. $I$ and $0$ mean the identity matrix and zero matrix, respectively, with compatible dimensions. $\mathbf{1}_n$ denotes the vector of dimension $N$ with all entries equal to one. In a symmetric matrix, symbol “$\sigma$” is used to denote the term that is induced by symmetry. $\otimes$ is the Kronecker product of matrices or vectors\cite{30}. We use $W^{-1}$ and $\lambda(W)$ to denote the inverse and the eigenvalues of any square matrix $W$, respectively. We use $W > 0$ to denote a positive-definite matrix $W$ with $\lambda_{\min}(W)$ and $\lambda_{\max}(W)$ being its minimum and maximum eigenvalues, respectively. Wherever the dimensions of the matrices are not mentioned, they are assumed to be of compatible dimensions.

We consider a DLMAS composed of $M$ coupled autonomous agents with a high-order linear dynamics:

$$\dot{x}_i(k + 1) = Ax_i(k) + Bu_i(k), \quad i = 1, \ldots, N \quad (1)$$

where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $x_i(k) \in \mathbb{R}^n$ and $u_i(k) \in \mathbb{R}^m$ are the state and the control input, respectively.

The construction of the control input $u_i$ depends on the available information from the $i$th agent and its neighbors. By saying agent $j$ is a neighbor of agent $i$, we mean that there exists a communication channel from $j$ to $i$. In this paper, we consider a time-varying set of the neighbors and use $\mathcal{N}_i(k)$ to denote the set of the neighbors of agent $i$ at time $k$. We call set $\mathcal{N}(k) = \{\mathcal{N}_i(k), i = 1, \ldots, N\}$ a switching information topology of DLMAS (1). It is well known that the information topology $\mathcal{N}(k)$ can be presented via a dynamic digraph. Letting a vertex set $V = \{1, \ldots, N\}$ represent the agents, and an time-varying edge set $E(k) \subseteq V \times V$ present the information topology $\mathcal{N}(k)$, i.e., $(i,j) \in E(k) \Leftrightarrow j \in \mathcal{N}_i(k)$, we get a dynamic digraph $G(k) = (V, E(k))$ of DLMAS (1).

We assume that the set of all possible information configurations of agent $i$ is $\{\mathcal{N}_i^l : l = 1, \ldots, M_i\}$, i.e., $\mathcal{N}_i(k) \in \{\mathcal{N}_i^l : l = 1, \ldots, M_i\}$ for any time $k$. Accordingly, DLMAS (1) admits $M$ information topologies $\{\mathcal{N}^l : l = 1, \ldots, M\}$, i.e., $\mathcal{N}(k) \in \{\mathcal{N}^l : l = 1, \ldots, M\}$. Letting $M = \{1, \ldots, M\}$, we define a switching signal $\sigma : \mathbb{Z}^+ \rightarrow M$ to describe the switching rule of the information topology $\mathcal{N}(k)$, i.e., $\mathcal{N}(k) = \mathcal{N}^\sigma(k) = l$, where $\mathbb{Z}^+$ is the set of positive integers.

Under the information topology $\mathcal{N}(k) \in \{\mathcal{N}^l : l \in M\}$, a control input (or a protocol) of agent $i$ is given by

$$u_i(k) = K_{\sigma(k)} \sum_{j \in \mathcal{N}_i(k)} w_{ij}^{\sigma(k)}(x_j(k-d(k)) - x_i(k-d(k))) \quad (2)$$

where $K_{\sigma(k)} \in \mathbb{R}^{m \times n}$ is a feedback matrix, $w_{ij}^{\sigma(k)}$ is a scalar weight under corresponding information topology $\mathcal{N}^\sigma$, where $i, j = 1, \ldots, N, l \in M$. The time-varying delay $d(k)$ is assumed to satisfy $d_1 \leq d(k) \leq d_2$, where $d_1$ and $d_2$ are constant positive scalars representing the lower and upper delays, respectively.

DLMAS (1) with protocol (2) has the following vector form:

$$\begin{align*}
x(k+1) &= (I_N \otimes A)x(k) - (I_{\sigma(k)} \otimes BK_{\sigma(k)})x(k-d(k)) \\
x(0) &= \varphi(\theta), \quad \forall \theta \in (-d_2, -d_2 + 1, \ldots, 0) \quad (3)
\end{align*}$$

where $x = [x_1^T, \ldots, x_N^T]^T$, $\varphi(\theta)$ is a given initial condition sequence. $I_{\sigma(k)}$ is the $N \times N$ weighted graph Laplacian induced by the information topology $\mathcal{N}(k)$ and its entry is defined by

$$t_{ij}^{\sigma(k)} = \begin{cases}
    \sum_{p \in \mathcal{N}_i(k)} w_{ip}^{\sigma(k)} & j = i \\
    -w_{ij}^{\sigma(k)} & j \neq i, j \in \mathcal{N}_i(k) \\
    0 & j \neq i, j \notin \mathcal{N}_i(k)
\end{cases} \quad (4)$$

Definition 1. Under the given information topologies $\{\mathcal{N}^l : l = 1, \ldots, M\}$ and the switching signal $\sigma$, DLMAS (1) with protocol (2) is said to achieve global state consensus if for any given initial condition sequence $\varphi(\theta)$, there exists an $n$-dimensional vector function $\xi(k, \varphi(\theta))$ depending on the initial sequence such that $\lim_{k \rightarrow \infty} \|x_i(k) - \xi(k, \varphi(\theta))\| = 0$. The function $\xi(k, \varphi(\theta))$ is called a state consensus function.

Next, we show that the global state consensus problem of DLMAS (1) with protocol (2) can be equivalently transformed into an asymptotic stability problem of a corresponding TDSLS via an appropriate linear transformation.

We select linear transformation matrix $T$ as

$$T = \begin{bmatrix}
    1 & -1 & 0 & \cdots & 0 \\
    0 & 1 & -1 & \cdots & 0 \\
    \vdots & \vdots & \ddots & \vdots & \vdots \\
    0 & 0 & 0 & 1 & -1 \\
    1 & 1 & 1 & \cdots & 1
\end{bmatrix} \otimes I_n = \begin{bmatrix}
    T_0 \\
    1_N
\end{bmatrix} \otimes I_n \quad (5)$$

and the corresponding inverse matrix $T^{-1}$ as

$$T^{-1} = \begin{bmatrix}
    N-1 & N-2 & \cdots & 1 & 1 \\
    -1 & N-2 & \cdots & 1 & 1 \\
    \vdots & \vdots & \ddots & \vdots & \vdots \\
    -1 & -2 & \cdots & 1 & 1 \\
    -1 & -2 & \cdots & -(N-1) & 1
\end{bmatrix} \otimes I_n = \begin{bmatrix}
    \tilde{T}_0 \\
    N^{-1}1_N \otimes I_n
\end{bmatrix} \quad (6)$$
Now we propose a linear transformation for system (3)
\[
\bar{x} = Tx
\]
and transform system (3) into the following system
\[
\bar{x}(k+1) = T(I_N \otimes A)T^{-1}\bar{x}(k) - \sum_{\sigma(k)} BK_{\sigma(k)}T^{-1}\bar{x}(k) + \bar{d}(k)
\]
\[
\bar{x}(0) = \varphi(\theta), \quad \forall \theta \in \{-d_1, -d_2, 1, \cdots, 0\}
\]
where \(\varphi(\theta) = T\varphi(\theta)\).

Letting \(\bar{x} = [y^T, z^T]^T\), where \(y = [x_1^T, \cdots, x_{N-1}^T]^T\) and \(z = \bar{x}_N\), we rewrite system (8) as the following two parts
\[
y(k+1) = \bar{A}y(k) + \bar{A}_{\sigma(k)}y(k-d(k)) + \bar{B}_1z(k) + \bar{B}_2z(k-d(k))
\]
\[
z(k+1) = \bar{C}y(k) + \bar{C}_{\sigma(k)}y(k-d(k)) + \bar{D}_1z(k) + \bar{D}_2z(k-d(k))
\]
where \(\bar{A} = I_{N-1} \otimes A\), \(\bar{A}_{\sigma(k)} = -\bar{T}_0L_{\sigma(k)}\bar{T}_0 \otimes BK_{\sigma(k)}\), \(\bar{B}_1 = \bar{B}_2 = 0\), \(\bar{C} = 0\), \(\bar{C}_{\sigma(k)} = -1^T_0L_{\sigma(k)}\bar{T}_0 \otimes BK_{\sigma(k)}\), \(\bar{D}_1 = \bar{A}\) and \(\bar{D}_2 = \bar{A}\).

**Definition 2**. The equilibrium point \(\bar{x} = 0\) of system (8) is said to be asymptotically stable with respect to the partial variable \(y\) (or briefly asymptotically \(y\)-stable) if for any \(\varepsilon > 0\), there is a \(\delta(\varepsilon)\) such that any perturbed trajectory \(\bar{x}(k) = [y^T(k), z^T(k)]^T\), starting from the initial sequence \(\varphi(\theta)\) limited by \(\sup_{\theta \leq 0} \|\varphi(\theta)\| \leq \delta(e)\) satisfies \(\sup_{k \geq 0} \|y(k)\| \leq \varepsilon\) and moreover \(\lim_{k \to \infty} \|y(k)\| = 0\).

In our case, because of \(\bar{B}_1 = \bar{B}_2 = 0\, the asymptotic stability of (8) is in fact equivalent to asymptotic stability of the following time-delayed switched linear system (TDSLs)
\[
y(k+1) = \bar{A}y(k) + \bar{A}_{\sigma(k)}y(k-d(k))
\]
\[
y(0) = \varphi(\theta), \forall \theta \in \{-d_2, -d_2 + 1, \cdots, 0\}
\]
where \(\varphi(y)\) is the first \(n(N-1)\)-dimensional components of the initial sequence \(\varphi(\theta)\).

**Definition 3**. TDSLs (10) is said to be asymptotically stable under switching signal \(\sigma\), if there exist some scalars \(K > 0\) and \(0 < \beta < 1\), such that the solution \(y(k)\) of TDSLs (10) satisfies \(|y(k)| < K \beta^{k-1} \sup_{\theta \leq 0} \|\varphi(\theta)\| \leq \varepsilon\), \(\forall k \geq k_0\), where \(\sup_{k \geq 0} \|y(k)\| = \sup_{k \geq -d_2} \|y(k)\|\). From the description above, we have the following lemma.

**Lemma 1.** Under the given information topologies \(\{N^i : i = 1, \cdots, M\}\) and the switching signal \(\sigma\), DLMAS (1) with protocol (2) achieves global state consensus if and only if TDSLs (10) is asymptotically stable.

Lemma 1 builds a bridge between the consensus problem of DLMAS (1) with protocol (2) and the asymptotic stability problem of the corresponding TDSLs (10).

For the sake of expression, we first introduce the following definition about the information topologies. This definition is similar to the concepts of controllability and stabilization in some sense.

**Definition 4.** An information topology is called to be consensusable for DLMAS (1) if under the information topology there is a protocol of the form of (2) such that DLMAS (1) with this protocol can achieve global state consensus. Otherwise, the information topology is called to be non-consensusable.

From Lemma 1, the consensusability of DLMAS (1) with the given information topology and the time-varying delay can be transformed to the asymptotic stability for the corresponding subsystem of TDSLs (10). Therefore, we can verify the consensusability for the given information topology by using the results for the asymptotic stability of discrete-time systems with time-varying state delay in [32].

In the paper, we consider two cases about the given information topologies \(\{N^i : i = 1, \cdots, M\}\). The first case is that each information topology in set \(\{N^i : i = 1, \cdots, M\}\) is consensusable. The second one is that only some of the information topologies in \(\{N^i : i = 1, \cdots, M\}\) are consensusable.

For both of the cases, we try to characterize the switching signals of the information topologies such that DLMAS (1) with protocol (2) achieves global state consensus. By Lemma 1, we use stability results of TDSLs to deduce average dwell-time conditions of the switching signals.

**2 Main results**

Now we introduce the definition of the average dwell-time and a useful lemma.

**Definition 5.** Let \(\sigma\) be a given switching signal. For any \(k > k_0\), let \(N_{\sigma}(k)\) denote the switching number of \(\sigma\) during \([k_0, k)\). If \(N_{\sigma}(k_0) \leq N_0 + (k - k_0)T_{ave}^{-1}\) holds for constants \(T_{ave} > 0\) and \(N_0 > 0\), then \(T_{ave}\) is called the average dwell-time and \(N_0\) the chatter bound. In this paper, we choose \(N_0 = 0\) and use \(S[T_{ave}]\) to denote the set of all switching signals satisfying \(N_{\sigma}(k_0) \leq (k - k_0)T_{ave}^{-1}\).

**Lemma 2.** For any constant matrix \(W = W^T \geq 0\), and two positive integers \(r\) and \(r_0\) satisfying \(r \geq r_0 > 1\), the following inequality holds
\[
\sum_{k=r_0}^{r} x^T(k)Wx(k) \leq \bar{r} \sum_{k=r_0}^{r} x^T(k)Wx(k) \leq \bar{r} \sum_{k=r_0}^{r} x^T(k)Wx(k)
\]
where \(\bar{r} = r - r_0 + 1\).

**2.1 Case 1:** all the information topologies are consensusable

In this subsection, we assume that each of the information topologies in set \(\{N^i : i = 1, \cdots, M\}\) is consensusable. We analyze the global state consensus of DLMAS (1) with protocol (2) by an average dwell-time scheme.

**Theorem 1.** For given constant scalars \(0 < \alpha < 1\), \(\mu \geq 1\) and any delay \(\delta(k)\) satisfying \(d_1 \leq \delta(k) \leq d_2\), if there exist positive-definite matrices \(P_i, Q_1i, Q_2i, Ri, R_2i, Ri, R_2i, i \in \mathcal{M}\), such that the following inequalities hold,

\[
\begin{bmatrix}
\Pi & d_1 \Phi_1^T R_1i & d_2 \Phi_2^T R_2i & \Phi_1^T P_i \\
* & -R_1i & 0 & 0 \\
* & * & -R_2i & 0 \\
* & * & * & -P_i \\
P_i & \mu P_i & Q_1i & \mu Q_2i \\
R_1i & \mu R_1i & R_2i & \mu R_2i \\
& & & & \forall i, j \in \mathcal{M}
\end{bmatrix} < 0
\]

\[
T_{ave} > T_{ave}^{-1} = \frac{\ln \mu}{\ln(1 - \alpha)}
\]

then for any switching signal \(\sigma \in S[T_{ave}]\) with the average dwell-time \(T_{ave}\) satisfying (14), DLMAS (1) with protocol (2) can achieve global state consensus, where

\[
\Phi_1 = [A_1i - \bar{A}_2i] \quad \Phi_2 = [A_1i - \bar{A}_2i] \quad 0
\]
with
\[
\Pi = \begin{bmatrix}
\Pi_{11} & 0 & \Pi_{13} & 0 \\
\ast & \Pi_{22} & \Pi_{23} & \Pi_{24} \\
\ast & \ast & \Pi_{33} & 0 \\
\ast & \ast & \ast & \Pi_{44}
\end{bmatrix}
\]

where
\[
\begin{align*}
\Pi_{11} &= Q_{11} + Q_{21} - (1 - \alpha)^{d_1} R_{11} \\
\Pi_{13} &= (1 - \alpha)^{d_1} R_{13} \\
\Pi_{22} &= -2(1 - \alpha)^{d_2} R_{21} \\
\Pi_{23} &= (1 - \alpha)^{d_1} R_{23} \\
\Pi_{24} &= (1 - \alpha)^{d_2} R_{24} \\
\Pi_{33} &= -(1 - \alpha)^{d_1} R_{13} - (1 - \alpha)^{d_2} R_{23} - (1 - \alpha)^{d_1} Q_{13} \\
\Pi_{44} &= -(1 - \alpha)^{d_2} Q_{23} - (1 - \alpha)^{d_2} R_{41}
\end{align*}
\]

**Proof.** Given a switching signal \(\Phi\), let \(\{k_l : l = 0, 1, \cdots\}\) be the corresponding sequence of switching instants. Suppose that \(\sigma(k_l) = i\) for \(k \in [k_l, k_{l+1})\), where \(i \in \mathcal{M}\). Choose the following Lyapunov functionals for TDSLs (10),
\[
V_i(k) = V_{i1}(k) + V_{i2}(k) + V_{i3}(k), \quad k \in [k_l, k_{l+1})
\]
where
\[
\begin{align*}
V_{i1}(k) &= y^T(k) P_i y(k) \\
V_{i2}(k) &= \sum_{s=k-d_1}^{k} (1 - \alpha)^{k-s-1} y^T(s) Q_{11} y(s) + \sum_{s=k-d_2}^{k} (1 - \alpha)^{k-s-1} y^T(s) Q_{22} y(s) \\
V_{i3}(k) &= d_1 \sum_{\theta = -d_1}^{k-1} \sum_{s = k + \theta}^{k-1} (1 - \alpha)^{k-s-1} \eta^T(s) R_{11} \eta(s) + d_2 \sum_{\theta = -d_2}^{k-1} \sum_{s = k + \theta}^{k-1} (1 - \alpha)^{k-s-1} \eta^T(s) R_{22} \eta(s) \eta(s) = y(s + 1) - y(s)
\end{align*}
\]

The forward difference \(\Delta V_i(k) = V_i(k+1) - V_i(k)\) of the Lyapunov functional \(V_i(k)\) along any trajectories of TDSLs (10) is given by
\[
\Delta V_{i1}(k) + \alpha V_{i1}(k) = \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \\]
According to (19) and (13), one can obtain that
\[
V_{e(k)}(k) \leq (1 - \alpha)^{k-k_i} V_{e(k_i)}(k_i) \\
(1 - \alpha)^{k-k_i} \mu V_{e(k_i-1)}(k_i) \leq \cdots \leq \\
(1 - \alpha)^{k-k_0} \mu^{k-k_0} V_{e(k_0)}(k_0) \\
((1 - \alpha)^{\mu^{k-k_0}})^{(k-k_0)} V_{e(k_0)}(k_0) \leq 
\] (21)

In addition, for the considered Lyapunov functional, it follows from (21) that
\[
\beta \|y(k)\|^2 \leq V_{e(k)}(k) \leq (1 - \alpha)^{\mu^{k-k_0}} V_{e(k_0)}(k_0) \leq \\
(1 - \alpha)^{\mu^{k-k_0}} \beta \|y(k_0)\|^2 \|1_i\|_1 
\] (22)

which yields \( \|y(k)\| \leq \sqrt{\beta_1/\beta_2} \beta^{(k-k_0)} \|y(k_0)\|_1 \), where
\[
\beta_1 = \min_{\forall i \in M} \min_{\forall i \in M} \lambda_{\min}(P_i) \\
\beta_2 = \max_{\forall i \in M} \lambda_{\max}(P_i) + d_i \max_{\forall i \in M} \lambda_{\max}(Q_{i1}) + \\
2d_i (d_i + 1) \max_{\forall i \in M} \lambda_{\max}(R_{i1}) + \\
2d_2 (d_2 + 1) \max_{\forall i \in M} \lambda_{\max}(R_{i2}) \\
\beta = \sqrt{1 - \alpha \mu^{k-k_0}} 
\]

Therefore, from condition (14), one can readily obtain \( \beta < 1 \). According to Definition 2, the considered TDSLS (10) is globally exponentially stable. Furthermore, from Lemma 1, DLMAS (1) with protocol (2) can achieve global state consensus for any switching signal \( \sigma \in S[T_{ave}] \) with the average dwell-time \( T_{ave} \) satisfying (14). \( \Box \)

2.2 Case 2: some of the information topologies are non-consensusable

In the previous subsection, we assumed that all the information topologies are consensusable. In this subsection, we consider a more general case, that is, the set of the information topologies consists of both consensusable and nonconsensusable ones. Without loss of generality, we use \( \mathcal{M}^- \) to denote the index subset of the consensusable information topologies and \( \mathcal{M}^+ \) the index subset of the nonconsensusable ones. Thus, \( \mathcal{M}^- \cup \mathcal{M}^+ = \mathcal{M} \).

In this case, the average dwell-time condition is no longer sufficient for the consensus of DLMAS (1). Let \( T^- (k) \) (respectively \( T^+ (k) \)) denote the total activation time of the consensusable information topologies (respectively the non-consensusable ones) up to time \( k \). Let \( \alpha^- \) and \( \alpha^+ \) be positive numbers which will be selected later. For any given \( \alpha \in (0, \alpha^-) \), we choose an arbitrary \( \alpha \in (0, \alpha^-) \) and propose the switching signals satisfying the following dwell-time ratio condition:
\[
\inf_{k \geq 0} \frac{T^- (k)}{T^+ (k)} \geq \frac{\ln (1 + \alpha^+) - \ln (1 - \alpha^-)}{\ln (1 - \alpha^+) - \ln (1 - \alpha^-)} 
\] (23)

Theorem 2. For given constant scalars \( \alpha^- \in (0,1) \), \( \alpha^+ > 0 \), \( \mu \geq 1 \) and any delay \( d(k) \) satisfying \( d_1 \leq d(k) \leq d_2 \), if there exist positive-definite matrices \( P_i, Q_{i1}, Q_{i2}, R_{i1}, R_{i2}, i \in \mathcal{M} \), such that the following equalities hold,
\[
\begin{bmatrix}
\Pi^- & d_2 \Phi_{i2}^2 R_{i1} & d_2 \Phi_{i2}^2 R_{i2} & \Phi_{i1}^T P_i \\
* & -R_{i1} & 0 & 0 \\
* & * & -R_{i2} & 0 \\
* & * & * & -P_i 
\end{bmatrix} < 0, \quad i \in \mathcal{M}^- 
\] (24)

\[
\begin{bmatrix}
\Pi^+ & d_1 \Phi_{i1}^2 R_{i1} & d_1 \Phi_{i1}^2 R_{i2} & \Phi_{i1}^T P_i \\
* & -R_{i1} & 0 & 0 \\
* & * & -R_{i2} & 0 \\
* & * & * & -P_i 
\end{bmatrix} < 0, \quad i \in \mathcal{M}^+ 
\] (25)

\[
R_{i1} \leq \mu R_{i1}, \quad R_{i2} \leq \mu R_{i2}, \quad \forall i, j \in \mathcal{M} 
\] (26)

\[
\frac{T_{ave}}{T} > \frac{\ln(1 - \alpha^-)}{\ln(1 - \alpha^+)} 
\] (27)

then for any switching signal \( \sigma \in S[T_{ave}] \) satisfying the average dwell-time condition (27) and dwell-time ratio condition (23), DLMAS (1) with protocol (2) can achieve global state consensus, where
\[
\Pi^- = \begin{bmatrix}
\Pi_{11}^- & 0 & 0 & 0 \\
* & \Pi_{13}^- & 0 & 0 \\
* & * & \Pi_{23}^- & 0 \\
* & * & * & \Pi_{44}^- 
\end{bmatrix} \\
\Pi^+ = \begin{bmatrix}
\Pi_{11}^+ & 0 & 0 & 0 \\
* & \Pi_{13}^+ & 0 & 0 \\
* & * & \Pi_{23}^+ & 0 \\
* & * & * & \Pi_{44}^+ 
\end{bmatrix} \\
\Phi_1 = [A_{i1}, \bar{A}_{i1}, 0, 0] \\
\Phi_2 = [A_{i1} - I, \bar{A}_{i1}, 0, 0]
\]

with
\[
\begin{align*}
\Pi_{11}^- &= Q_{i1} + Q_{i2} - (1 - \alpha^-)^{d_1} R_{i1} \\
\Pi_{13}^- &= (1 - \alpha^-)^{d_1} R_{i1} \\
\Pi_{22}^- &= -2(1 - \alpha^-)^{d_2} R_{i2} \\
\Pi_{23}^- &= (1 - \alpha^-)^{d_2} R_{i2} \\
\Pi_{24}^- &= (1 - \alpha^-)^{d_2} R_{i2} \\
\Pi_{33}^- &= -(1 - \alpha^-)^{d_2} R_{i1} - (1 - \alpha^-)^{d_2} R_{i2} - \\
&\quad (1 - \alpha^-)^{d_1} Q_{i1} \\
\Pi_{44}^- &= -(1 - \alpha^-)^{d_2} Q_{i1} - (1 - \alpha^-)^{d_2} R_{i2} \\
\Pi_{11}^+ &= Q_{i1} + Q_{i2} - (1 + \alpha^+)^{d_1} R_{i1} \\
\Pi_{13}^+ &= (1 + \alpha^+)^{d_1} R_{i1} \\
\Pi_{22}^+ &= -2(1 + \alpha^+)^{d_2} R_{i2} \\
\Pi_{23}^+ &= (1 + \alpha^+)^{d_2} R_{i2} \\
\Pi_{24}^+ &= (1 + \alpha^+)^{d_2} R_{i2} \\
\Pi_{33}^+ &= -(1 + \alpha^+)^{d_2} R_{i1} - (1 + \alpha^+)^{d_2} R_{i2} - \\
&\quad (1 + \alpha^+)^{d_1} Q_{i1} \\
\Pi_{44}^+ &= -(1 + \alpha^+)^{d_2} Q_{i1} - (1 + \alpha^+)^{d_2} R_{i2} \\
\end{align*}
\]

Proof. Suppose that \( \sigma(k) = i \) for \( k \in [k_i, k_{i+1}) \), where \( i \in \mathcal{M} \). For \( i \in \mathcal{M}^- \), select the piecewise Lyapunov functionals (15) by replacing \( \alpha \) by \( \alpha^- \). Whereas for \( i \in \mathcal{M}^+ \),
select the following piecewise Lyapunov functionals,
\[ V_i(k) = V_{i1}(k) + V_{i2}(k) + V_{i3}(k), \quad k \in [k_i, k_{i+1}) \]  
(28)
where
\[ V_{i1}(k) = y_T(k)P_i y(k) \]
\[ V_{i2}(k) = \sum_{s = k-d_1_{i}}^{k-1} (1 + \alpha^+)^{k-s-1} y_T(s)Q_i y(s) + \]
\[ \sum_{s = k-d_2_{i}}^{k-1} (1 + \alpha^-)^{k-s-1} y_T(s)Q_{2i} y(s) \]
\[ V_{i3}(k) = d_i \sum_{s = k}^{k-1} (1 + \alpha^+)^{k-s-1} \eta^T(s)R_i \eta(s) + \]
\[ D_2 \sum_{s = k-d_4_{i}}^{k-1} (1 + \alpha^-)^{k-s-1} \eta^T(s)R_{2i} \eta(s) \]
\[ \eta(s) = y(s+1) - y(s) \]

Similar to (16) ~ (19), when \( k \in [k_i, k_{i+1}) \), we have
\[ V(k) := V_{e}(k) = V_i(k) \leq \]
\[ \begin{cases} (1 - \alpha^-)^{k-k_i} V_{e}(k_i)(k_i), & i \in M^- \\ (1 + \alpha^+)^{k-k_i} V_{e}(k_i)(k_i), & i \in M^+ \end{cases} \]  
(29)
Using (29) and \( N_s(k_i, k) \leq (k - k_0) / T_{\text{ave}} \) leads to
\[ V(k) \leq \mu^{N_s(k_0, k)} (1 + \alpha^+)^{T^+(k)}(1 - \alpha^-)^{T^-(k)} V_{e}(k_0)(k_0) \leq e^{(k-k_0)/T_{\text{ave}} + T^+(k)(1+\alpha^+) + T^-(k)(1-\alpha^-)} \times \]
\[ V_{e}(k_0) \]  
(30)
Combining (23), (27) and (30), we get
\[ V(k) \leq e^{(\ln(1+\alpha^+) + \ln \mu / T_{\text{ave}})(k-k_0)} V_{e}(k_0)(k_0) = \]
\[ \mu^{(k-k_0)/T_{\text{ave}}} (1 - \alpha^-)^{(k-k_0)} V_{e}(k_0)(k_0) \leq \]
\[ ((1 - \alpha)\mu^{1/T_{\text{ave}}})^{(k-k_0)} V_{e}(k_0)(k_0) \]  
(31)
Therefore, we have
\[ \|y(k)\| \leq \sqrt{\beta_1 / \beta_2} \beta^2(k-k_0) \|y(k_0)\|, \]
where \( \beta_1, \beta_2 \) and \( \beta \) are defined the same as in Theorem 1. This implies that TDSLS (10) is exponentially stable for any switching signal satisfying (23) and (27). Thus, we can get that DLMAS (1) with protocol (2) can achieve global state consensus.

As also shown in [26], the switching condition (23) is quite easy to satisfy. For example, we can first activate the consensusable information topologies with a time period of about \( 2\ln(1+\alpha^+) - \ln(1-\alpha^-) | T_0 \), and then activate the nonconsensusable ones with a time period \( \ln(1-\alpha^-) - \ln(1-\alpha^-) / T_0 \), where \( T_0 \) is a positive time unit sufficiently large to satisfy the average dwell-time condition (23), and \( \lceil \cdot \rceil \) denote the upper integer bound and lower integer bound, respectively.

In this paper, we verify the average dwell-time condition for the DLMAS to achieve consensus with directed switching information topologies and time-varying delay. Furthermore, according to Theorem 1 and Theorem 2, we can design switching sinal for the DLMAS. The algorithm is as follows.

Algorithm 1.

Step 1. Give the dynamics of the DLMAS, the information topologies, the connected weights, the feedback matrices, the lower and upper delay.

Step 2. Using linear transformation matrix \( T \) in (5) and the corresponding inverse matrix \( T^{-1} \) in (6), one gets TDSLS (10).

Step 3. Judge the cases of the information topologies and select \( \alpha^- \), \( \alpha^+ \) and \( \mu \).

Step 4. Calculate the positive-definite matrices \( P_i, Q_{1i}, Q_{2i}, R_{ii}, R_{2i}, i \in M \).

Step 5. Select \( \alpha \) and get \( T_{\text{ave}} \).

Step 6. Get the dwell-time ratio according to (23).

Step 7. Design switching signal \( \sigma \in S[T_{\text{ave}}] \) satisfying the average dwell-time (27). Thus DLMAS (1) with protocol (2) can achieve global state consensus.

Notice that for the case that all the information topologies are consensusable, one need not select parameter \( \alpha^- \) but uses \( \alpha \) instead of \( \alpha^- \) in Step 3, and omits Step 6.

3 Numerical examples

Now we give numerical examples to illustrate the theoretical results obtained in the previous section.

We consider a DLMAS consisting of 6 agents, and the ith agent has the dynamics:
\[ x_i(k+1) = \begin{bmatrix} 1 & 0 & 0 & -0.1 & -0.3 & -0.2 \\ -0.5 & 0.5 & 0 & 0 & 0 & 0 \\ -0.3 & -0.2 & 0.5 & 0 & 0 & 0 \\ -0.5 & 0 & 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.4 & 0.6 & -0.2 \\ 0 & 0 & 0 & 0 & -0.3 & 0.3 \end{bmatrix} u_i(k) \]  
(32)
where \( i = 1, \ldots, 6 \), \( x_i(k) = [p_i(k), v_i(k)]^T \), \( p_i(k) \in \mathbb{R} \) is the position state, \( v_i(k) \in \mathbb{R} \) is the velocity state, and \( u_i(k) \in \mathbb{R} \) is the protocol. The time-varying delay is assumed to satisfy \( 2 \leq d(k) \leq 4 \).

Case 1. All the information topologies are consensusable.

Fig. 1 shows the information topologies of the 6 agents. The corresponding weighted graph Laplacians are as follows:

![Fig. 1 Two information topologies](image-url)

Select the feedback matrix \( K_1 = [0.1, 1.1] \) for Fig. 1 (a) and \( K_2 = [0.028, 1] \) for Fig. 1 (b), respectively. The corresponding TDSLS (10) are all asymptotically stable according to the result in [32], i.e., DLMAS (32) with two information topologies in Fig. 1 are all consensusable. Choosing \( \alpha = 0.0029, \mu = 3.2 \), we get the positive definite matrix \( P_i, Q_{1i}, Q_{2i}, R_{ii}, R_{2i}, i = 1, 2 \) and \( T_{\text{ave}} > T_{\text{ave}}^* = 232.0481 \). The position and velocity trajectories starting from the initial position \([10, 100, 200, 300, 400, 500]^T\) and initial velocity...
[12, 10, 8, 6, 4, 2]^T with sampling period 0.1 s are shown in Figs. 2 (a) and (b), respectively. Fig. 3 shows the switching signal \( \sigma \in S[T_{\text{ave}}] \) with \( T_{\text{ave}} = 235 \) under all consensusable information topologies.

By selecting the same feedback matrix \( K_1 \) for Fig. 4 (a) as for Fig. 1 (a), DLMAS (32) with information topology Fig. 4 (a) is consensusable with information topology Fig. 4 (a). Then by selecting the feedback matrix \( K_2 = [0, 0.01, 1] \) for Fig. 4 (b), TDSLS (10) is unstable according to the result in [32], i.e., DLMAS (32) with protocol (2) is nonconsensusable with information topology of Fig. 4 (b). Choosing \( \alpha^- = 0.0115, \alpha^+ = 0.006, \alpha^* = 0.01, \) and \( \mu = 4 \), we get the positive definite matrix \( P, Q_1, Q_2, R_1, R_2, t = 1.2 \), and \( T_{\text{ave}} > T_{\text{ave}} = 137.9351 \). According to condition (23), we obtain inf \( T_{\sigma_k} \geq 2.6593 \), and choose 3 for it.

In order to satisfy both the switching law (23) and the average dwell-time condition (27), we choose to activate nonconsensusable information topology and consensusable information topology with time steps 69 and 207, respectively. The position and velocity trajectories of the above DLMAS (32) with protocol (2) are shown in Figs. 5 (a) and (b), respectively, where the initial position state, and the initial velocity state, and the sampling period are the same as Case 1. Fig. 6 shows the switching signal \( \sigma \in S[T_{\text{ave}}] \) with \( T_{\text{ave}} = 138 \) under some consensusable information topologies.

Fig. 4 Two information topologies

Case 2. Some of the information topologies are consensusable.

Fig. 4 shows another switching information topologies of the 6 agents. The weighted graph Laplacian for Fig. 4 (a) is the same as Fig. 1 (a), and the weighted graph Laplacian for Fig. 4 (b) is as follows:

\[
L_2 = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0.05 & -0.05 & 0 & 0 & 0 & 0 \\
-0.2 & 0 & 0.5 & 0 & -0.3 & 0 \\
0 & -0.6 & 0 & 0.6 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -0.4 & 0 & 0 & 0.4
\end{bmatrix}
\]
4 Conclusion

In this paper, we have considered the global state consensus problem of high-order DLMAs with directed switching information topologies and varying time-delay. We used a linear transformation to change the consensus problem with switching directed information topology and varying time-delay into an asymptotical stability problem of corresponding TDSLS equivalently. For the characterization of the switching signals of the information topologies we proposed the average dwell-time conditions for two cases: 1) all the information topologies are consensusable; 2) some of the information topologies are consensusable. Numerical examples have shown the effectiveness of the proposed results.

References


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