Sufficient and Necessary Condition of Admissibility for Fractional-order Singular System

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Abstract This paper focuses on the admissibility condition for fractional-order singular system with order $\alpha \in (0,1)$. The definitions of regularity, impulse-free and admissibility are given first, then a sufficient and necessary condition of admissibility for fractional-order singular system is established. A numerical example is included to illustrate the proposed condition.

Key words Fractional-order singular systems, regularity, impulse-free, admissibility

Citation Yu Yao, Jiao Zhuang, Sun Chang-Yin. Sufficient and necessary condition of admissibility for fractional-order singular system. *Acta Automatica Sinica*, 2013, **39**(12): 2160−2164

DOI 10.3724/SP.J.1004.2013.02160

The history of fractional calculus is more than 300 years old. Classical works on the fractional integral and fractional differential equations have been done by Oldham et al.[1], and Poldubny[2]. Recently, increasing attentions are paid to fractional differential equations and their applications in various science and engineering fields, such as electrochemistry[3], electrode-electrolyte polarization[4], viscous damping[5−6], viscoelastic systems[7], electric fractal networks[8], electromagnetic waves[9] and so on.

As an important application of fractional calculus, fractional-order control systems[10] have attracted more and more interests in the last several years, for instance, some important issues of fractional-order systems, such as modeling, stability, controllability, observability were considered in [11−12]; $\mathrm{PI}^{\lambda}\mathrm{D}^{\mu}$ controller, the generalization of PID controller was proposed in [13]; CRONE control[14] was the first robust control method based on fractional differentiation for linear systems; robust stability of interval uncertain fractional-order linear time invariant systems was investigated in [15−17]; a numerical algorithm for stability testing of fractional-order delay systems is presented in [18]. For more knowledge about fractional-order control theory and its applications, one can refer to [19−24].

Singular systems[25] have been extensively studied in the past few decades due to the fact that singular systems can describe real physical systems better and more directly than regular systems. Naturally, many theoretical results for regular systems have been extended to singular cases[26]. It is well known that issues of concern for singular systems are much more complicated than those for regular systems, because we need to consider not only stability, but also regularity and the absence of impulses at the same time[27−28] for singular systems. Regularization and stabilization for singular fractional-order systems with order between 1 and 2 was first investigated in [29]. Admissibility is a very important property for singular systems, and to the best of our knowledge, there exist no results about admissibility for fractional-order singular system, so in this paper we give the sufficient and necessary condition of admissibility for fractional-order singular system with fractional order $\alpha \in (0,1)$.

This paper is organized as follows. Some preliminaries about fractional-order calculus and fractional-order systems are recalled in Section 1. The main result for fractional-order singular system with order $\alpha \in (0,1)$ is given in Section 2, which includes the extensions of some of the basic results of singular integer-order systems to fractional-order singular system, e.g., the definition of regularity, impulse-free and admissibility. In particular, a sufficient and necessary condition of admissibility for fractional-order singular system is given. In Section 3, a numerical example is included to illustrate the main result established in this paper.

1 Preliminaries

Fractional calculus[2] has become a powerful mathematical tool and is playing a more and more important role in modern science and engineering.

In order to utilize fractional calculus for the discussion in current paper, the fundamental definition of fractional calculus is recalled. The uniform formula of fractional-order integral is defined as follows:

$$ {}_0\mathrm{D}_t^{-\alpha} f(t) := \mathrm{D}^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)\mathrm{d}\tau $$

where $\alpha > 0$, $f(t)$ is an arbitrary integrable function, $\Gamma(\cdot)$ is Gamma function.

Based on the definition of fractional-order integral, the well known definition of Caputo fractional-order derivative operator is defined as:

$$ {}_0^C\mathrm{D}_t^{\alpha} f(t) := \mathrm{D}^{\alpha} f(t) = \tfrac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau)\mathrm{d}\tau $$

where $n-1 < \alpha < n$.

To proceed the discussion of the main result, the following lemma is given.

Lemma 1[17]**.** A fractional-order system: $\mathrm{D}^{\alpha}x(t) = Ax(t)$ $(0<\alpha<1)$ is asymptotically stable if and only if there exist two real symmetric positive definite matrices Q_{11} and Q_{21}, and two skew-symmetric matrices Q_{12} and Q_{22} such that

$$ \sum_{i=1}^{2}\sum_{j=1}^{2} sym\left\{\Theta_{ij} \otimes (Q_{ij}A)\right\} < 0 $$

$$ \begin{bmatrix} Q_{11} & Q_{12} \\ -Q_{12} & Q_{11} \end{bmatrix} > 0, \quad \begin{bmatrix} Q_{21} & Q_{22} \\ -Q_{22} & Q_{21} \end{bmatrix} > 0 $$

where $sym\{X\} := X^{\mathrm{T}} + X$, Θ_{ij} $(i,j=1,2)$ are defined as follows:

$$ \Theta_{11} = \begin{bmatrix} \sin\left(\frac{\pi\alpha}{2}\right) & -\cos\left(\frac{\pi\alpha}{2}\right) \\ \cos\left(\frac{\pi\alpha}{2}\right) & \sin\left(\frac{\pi\alpha}{2}\right) \end{bmatrix} \tag{1} $$

$$ \Theta_{12} = \begin{bmatrix} \cos\left(\frac{\pi\alpha}{2}\right) & \sin\left(\frac{\pi\alpha}{2}\right) \\ -\sin\left(\frac{\pi\alpha}{2}\right) & \cos\left(\frac{\pi\alpha}{2}\right) \end{bmatrix} \tag{2} $$

$$ \Theta_{21} = \begin{bmatrix} \sin\left(\frac{\pi\alpha}{2}\right) & \cos\left(\frac{\pi\alpha}{2}\right) \\ -\cos\left(\frac{\pi\alpha}{2}\right) & \sin\left(\frac{\pi\alpha}{2}\right) \end{bmatrix} \tag{3} $$

Manuscript received December 12, 2012; accepted May 13, 2013
Supported by National Outstanding Youth Science Foundation (61125306), Major Research Plan of National Natural Science Foundation of China (91016004), National Natural Science Foundation (61203071), Fundamental Research Funds for the Central Universities (FRF-TP-13-017A), and Specialized Research Fund for the Doctoral Program of Higher Education (20130006120027, 20110092110020)
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$$\Theta_{22} = \left[\begin{array}{cc} -\cos\left(\frac{\pi\alpha}{2}\right) & \sin\left(\frac{\pi\alpha}{2}\right) \\ -\sin\left(\frac{\pi\alpha}{2}\right) & -\cos\left(\frac{\pi\alpha}{2}\right) \end{array} \right] \tag{4}$$

2 Main result

Consider the following fractional-order singular (FOS) system:

$$E\mathrm{D}^{\alpha}x(t) = Ax(t),\ 0 < \alpha < 1 \tag{5}$$

where $A \in \mathbf{R}^{n\times n}$, $E \in \mathbf{R}^{n\times n}$ is a singular matrix such that $\mathrm{rank}(E) = r < n$, D^{α} denotes the Caputo derivative operator.

Parallel to integer order singular systems, the concerning basic definitions and relevant facts for FOS system (5) are given as follows.

Definition 1. For FOS system (5), the triplet (E, A, α) is called regular if there exists a constant scalar $c_0 \in \mathbf{C}$ such that $|c_0^{\alpha}E - A| \neq 0$, e.g., the pseudo-polynomial $|s^{\alpha}E - A|$ is not identically zero.

Similar to the proof of regularity of integer order singular systems, the regularity condition for fractional-order singular system is given as following.

Lemma 2. The triplet (E, A, α) in FOS system (5) is regular if and only if there exist two nonsingular matrices Q and P such that

$$QEP = \mathrm{diag}\left(I_{n_1}, N\right),\quad QAP = \mathrm{diag}\left(A_1, I_{n_2}\right) \tag{6}$$

where $n_1+n_2 = n$, $A_1 \in \mathbf{R}^{n_1\times n_1}$, $N \in \mathbf{R}^{n_2\times n_2}$ is nilpotent.

Proof. Sufficiency. Assume there exist two nonsingular matrices Q and P satisfying (5), then we can choose $c_0^{\alpha} \notin \sigma(A_1)$, where $\sigma(A_1)$ denotes the set of eigenvalues of A_1, then

$$\begin{aligned} |c_0^{\alpha}E - A| = \left|Q^{-1}\right|\left|P^{-1}\right|\left|c_0^{\alpha}QEP - QAP\right| = \\ \left|Q^{-1}\right|\left|P^{-1}\right|\left|c_0^{\alpha}I_{n_1} - A_1\right|\left|c_0^{\alpha}N - I_{n_2}\right| \neq 0 \end{aligned}$$

Thus (E, A, α) is regular.

Necessity. If (E, A, α) is regular, then there exists $c_0 \in$ C such that $|c_0^{\alpha}E - A| \neq 0$. Note that $\bar{E} = (c_0^{\alpha}E - A)^{-1}E$, $\bar{A} = (c_0^{\alpha}E - A)^{-1}A$, so it is easy to obtain $\bar{A} = c_0^{\alpha}\bar{E} - I$.

On the other hand, it follows from the Jordan canonical form decomposition that there exists a nonsingular matrix T such that $T\bar{E}T^{-1} = \mathrm{diag}\left(\bar{E}_1, \bar{E}_2\right)$, where $\bar{E}_1 \in \mathbf{R}^{n_1\times n_1}$ is nonsingular, $\bar{E}_2 \in \mathbf{R}^{n_2\times n_2}$ is nilpotent. From the above analysis, we know that $I - c_0^{\alpha}\bar{E}_2$ is nonsingular.

Let $Q = \mathrm{diag}\left(\bar{E}_1^{-1}, \left(c_0^{\alpha}\bar{E}_2 - I\right)^{-1}\right)T(c_0^{\alpha}E - A)^{-1}$, and $P = T^{-1}$. From $\bar{A} = c_0^{\alpha}\bar{E} - I$, one has

$$QEP = \mathrm{diag}\left(I_{n_1}, N\right), QAP = \mathrm{diag}\left(A_1, I_{n_2}\right)$$

where $A_1 = \bar{E}_1^{-1}\left(c_0^{\alpha}\bar{E}_2 - I\right)$, $N = \left(c_0^{\alpha}\bar{E}_2 - I\right)^{-1}\bar{E}_2$ is nilpotent.

Assume the triplet (E, A, α) in FOS system (5) is regular, then based on Lemma 2, the FOS system (5) can be transformed into

$$\left\{ \begin{array}{l} \mathrm{D}^{\alpha}x_1(t) = A_1x_1(t) \\ N\mathrm{D}^{\alpha}x_2(t) = x_2(t) \end{array} \right. \quad 0 < \alpha < 1 \tag{7}$$

where $\left[\ x_1(t) \quad x_2(t)\ \right]^{\mathrm{T}} = P^{-1}x(t)$, the initial state response of FOS system (7) is

$$\left[\begin{array}{c} x_1(t) \\ x_2(t) \end{array} \right] = P\left[\begin{array}{c} E_{\alpha,1}(A_1t^{\alpha})x_1(0) \\ -\sum\limits_{k=1}^{h-1}\delta^{((k-1)\alpha)}(t)N^kx_2(0) \end{array} \right],\quad t \geq 0 \tag{8}$$

where $x_1(t) \in \mathbf{R}^{n_1}$, $x_2(t) \in \mathbf{R}^{n_2}$, $n_1 + n_2 = n$, $\delta(t)$ is the impulse function, $E_{\alpha,\beta}(t)$ is the two-parameter Mittag-Leffler function[2].

From (8), we know that the triplet (E, A, α) is impulse-free if $N = 0$. □

Definition 2. The finite eigenvalues of $(\lambda E-A)$ in FOS system (5) are called finite dynamic modes of the triplet (E, A, α).

Let $\sigma(E, A, \alpha) = \{\lambda\,|\,\lambda \in \mathbf{C}, \lambda \text{ finite}, |\lambda \mathrm{E} - \mathrm{A}| = 0\}$ denotes the finite pole set for FOS system (5). It can be easily known from [11] that the FOS system (5) is asymptotically stable, if all the finite dynamic modes lie in the domain $\mathrm{D}_s^{\alpha} := \{\lambda\,|\,|\arg(\lambda)| > \alpha\pi/2, \lambda \in \mathbf{C}\}$. Stable regions for FOS system (5) of order $0 < \alpha < 1$, $\alpha = 1$ and $1 < \alpha < 2$ are illustrated in Fig. 1.

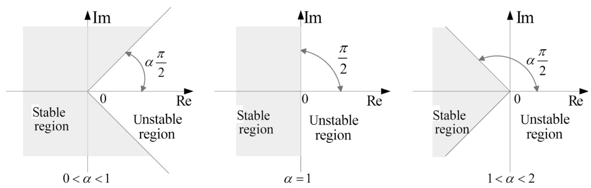

Fig. 1 Stable regions D_s^{α} for $0 < \alpha < 1$, $\alpha = 1$ and $1 < \alpha < 2$

Definition 3. The generalized eigenvectors ν satisfying $E\nu = 0$ are defined as:

1) The infinite eigenvector of order 1 satisfies $E\nu_i^1 = 0$.

2) The infinite eigenvector of order k satisfies $E\nu_i^k = A\nu_i^{k-1}, k > 1$.

Remark 1. Suppose that $E\nu^1 = 0$, then the infinite eigenvalues associated with the generalized principal vectors ν^k satisfying $E\nu^k = \nu^{k-1}$ are impulsive modes. The triplet (E, A, α) is impulse-free if and only if there exists no infinite eigenvector of order 2, ν^2.

Definition 4. FOS system (5) is said to be admissible, if the triplet (E, A, α) is regular, impulse-free, and all the finite eigenvalues of triplet (E, A, α) lie in the stable regions of D_s^{α}.

In the following, as the main result, a sufficient and necessary condition of admissibility for FOS system (5) is derived.

Theorem 1. Assume the triplet (E, A, α) is regular, then FOS system (5) is admissible, if and only if there exist two real symmetric positive definite matrices Q_{11} and Q_{21}, and two skew-symmetric matrices Q_{12} and Q_{22}, and $Q \in \mathbf{R}^{(n-r)\times n}$ such that

$$sym\left\{\ \sum_{i=1}^{2}\sum_{j=1}^{2}\left\{\Theta_{ij} \otimes \left(E^{\mathrm{T}}Q_{ij}A\right)\right\} + I_2 \otimes \left(Q^{\mathrm{T}}E_0^{\mathrm{T}}A\right)\ \right\} < 0 \tag{9}$$

where $\mathrm{rank}\ (E) = r < n$, $E_0 \in \mathbf{R}^{n\times(n-r)}$ is a matrix of full column rank such that $E^{\mathrm{T}}E_0 = 0$, $sym\{X\} := X^{\mathrm{T}} + X$, $\Theta_{ij}\ (i, j = 1, 2)$ are defined in (1) ~ (4).

Proof. Sufficiency. Proof by contradiction, assume that triplet (E, A, α) is impulsive, then it can be known similarly from Remark 1 that there exists an infinite eigenvector of order 2, $\nu^2 \in \mathbf{R}^n$ such that $E\nu^2 = A\nu^1$ and $E\nu^1 = 0$. By premultiplying $\left(I_2 \otimes \nu^1\right)^{\mathrm{T}}$ and postmultiplying $\left(I_2 \otimes \nu^1\right)$, then (9) becomes

$$\begin{aligned} &\left(I_2 \otimes \nu^1\right)^{\mathrm{T}} \times \\ &sym\left\{\ \sum_{i=1}^{2}\sum_{j=1}^{2}\left\{\Theta_{ij} \otimes \left(E^{\mathrm{T}}Q_{ij}A\right)\right\} + I_2 \otimes \left(Q^{\mathrm{T}}E_0^{\mathrm{T}}A\right)\ \right\} \times \\ &\left(I_2 \otimes \nu^1\right) < 0 \end{aligned}$$

Then one has

$$\begin{aligned}&I_2\otimes\left(\nu^1\right)^{\mathrm{T}}\times\\&sym\left\{\ \sum_{i=1}^2\sum_{j=1}^2\left\{\Theta_{ij}\otimes\left(E^{\mathrm{T}}Q_{ij}A\right)\right\}\ \right\}\left(I_2\otimes\nu^1\right)+\\&\quad I_2\otimes\left(\nu^1\right)^{\mathrm{T}}sym\left\{\ I_2\otimes\left(Q^{\mathrm{T}}E_0^{\mathrm{T}}A\right)\ \right\}\left(I_2\otimes\nu^1\right)<0\end{aligned}$$

Then

$$\begin{aligned}&I_2\otimes sym\left\{\ \sum_{i=1}^2\sum_{j=1}^2\left\{\Theta_{ij}\otimes\left(\nu^1\right)^{\mathrm{T}}\left(E^{\mathrm{T}}Q_{ij}A\right)\nu^1\right\}\ \right\}+\\&\quad I_2\otimes sym\left\{\ I_2\otimes\left(\nu^1\right)^{\mathrm{T}}\left(Q^{\mathrm{T}}E_0^{\mathrm{T}}A\right)\nu^1\ \right\}<0\end{aligned}$$

So, one has

$$\begin{aligned}&I_2\otimes sym\left\{\ \sum_{i=1}^2\sum_{j=1}^2\left\{\Theta_{ij}\otimes\left(E\nu^1\right)^{\mathrm{T}}(Q_{ij}A)\nu^1\right\}\ \right\}+\\&\quad I_2\otimes sym\left\{\ I_2\otimes\left(\nu^1\right)^{\mathrm{T}}\left(Q^TE_0^{\mathrm{T}}\right)A\nu^1\ \right\}<0\end{aligned}$$

Then

$$I_2\otimes sym\left[I_2\otimes\left(\nu^1\right)^{\mathrm{T}}\left(Q^{\mathrm{T}}E_0^{\mathrm{T}}\right)E\nu^2\right]<0$$

Then the following inequality holds

$$I_2\otimes\left[\left(\nu^1\right)^{\mathrm{T}}\left(Q^{\mathrm{T}}E_0^{\mathrm{T}}E\right)\nu^2+\left(\nu^2\right)^{\mathrm{T}}\left(E^{\mathrm{T}}E_0Q\right)\nu^1\right]<0$$

Of course, the above inequality does not hold, so the assumption is false. Then it can be known that triplet (E,A,α) is impulse-free.

Let λ be any finite eigenvalue of triplet (E,A,α), ν is the corresponding eigenvector, i.e., $A\nu=\lambda E\nu$ and $\nu^*A^{\mathrm{T}}=\bar{\lambda}\nu^*E^{\mathrm{T}}$. From (9), one has

$$\begin{aligned}&(I_2\otimes\nu)^*sym\left\{\begin{array}{l}\sum\limits_{i=1}^2\sum\limits_{j=1}^2\left\{\Theta_{ij}\otimes\left(E^{\mathrm{T}}Q_{ij}A\right)\right\}+\\I_2\otimes\left(Q^{\mathrm{T}}E_0^{\mathrm{T}}A\right)\end{array}\right\}(I_2\otimes\nu)=\\&\quad\sum_{i=1}^2\sum_{j=1}^2\left\{\Theta_{ij}\otimes\left(\lambda\nu^*E^{\mathrm{T}}Q_{ij}E\nu\right)\right\}+\\&\quad\sum_{i=1}^2\sum_{j=1}^2\left\{\Theta_{ij}^{\mathrm{T}}\otimes\left(\bar{\lambda}\nu^*E^{\mathrm{T}}Q_{ij}^{\mathrm{T}}E\nu\right)\right\}=\\&\quad\sum_{i=1}^2\sum_{j=1}^2\\&\quad\left(I_2\otimes\nu^*E^{\mathrm{T}}\right)\left\{\Theta_{ij}\otimes(\lambda Q_{ij})+\Theta_{ij}^{\mathrm{T}}\otimes\left(\bar{\lambda}Q_{ij}^{\mathrm{T}}\right)\right\}(I_2\otimes E\nu)=\\&\quad\left(I_2\otimes\nu^*E^{\mathrm{T}}\right)\left\{\sum_{i=1}^2\sum_{j=1}^2\left\{\Theta_{ij}\otimes(\lambda Q_{ij})+\Theta_{ij}^{\mathrm{T}}\otimes\left(\bar{\lambda}Q_{ij}^{\mathrm{T}}\right)\ \right\}\right\}\times\\&\quad(I_2\otimes E\nu)<0\end{aligned}$$

It is known from the above inequality that

$$\sum_{i=1}^2\sum_{j=1}^2\left\{\Theta_{ij}\otimes(\lambda Q_{ij})+\Theta_{ij}^{\mathrm{T}}\otimes\left(\bar{\lambda}Q_{ij}^{\mathrm{T}}\right)\right\}<0$$

Based on Lemma 4, one knows that both λ and $\bar{\lambda}$ lie in D^{α}, i.e., all the finite eigenvalues of triplet (E,A,α) lie in D^{α}, then FOS system (5) is admissible.

Necessities. As triplet (E,A,α) is regular and impulse-free, then there exist nonsingular matrices M and N, such that

$$MEN=\left[\begin{array}{c}M_1\\M_2\end{array}\right]E\left[\begin{array}{cc}N_1&N_2\end{array}\right]=\left[\begin{array}{cc}I_r&0\\0&0\end{array}\right]$$

$$MAN=\left[\begin{array}{c}M_1\\M_2\end{array}\right]A\left[\begin{array}{cc}N_1&N_2\end{array}\right]=\left[\begin{array}{cc}A_1&0\\0&I_{n-r}\end{array}\right]$$

where $M_1\in\mathbf{R}^{r\times n}$, $N_1\in\mathbf{R}^{n\times r}$.

Note that $M_2AN=\left[\begin{array}{cc}0&I_{n-r}\end{array}\right]$, $M_2E=0$, then it can be known from Lemma 1 and the D^{α}-stability of triplet (E,A,α) that there exist two real symmetric positive definite matrices $\bar{Q}_{11}$ and $\bar{Q}_{21}$, and two skew-symmetric matrices $\bar{Q}_{12}$ and $\bar{Q}_{22}$, such that

$$\sum_{i=1}^2\sum_{j=1}^2 sym\left\{\Theta_{ij}\otimes\left(\bar{Q}_{ij}A_1\right)\right\}<0$$

Therefore, there exists a sufficiently small positive constant ε, such that

$$\sum_{i=1}^2\sum_{j=1}^2 sym\left\{\Theta_{ij}\otimes\left(\bar{Q}_{ij}A_1\right)\right\}+I_2\otimes\left(\frac{\varepsilon}{2}N_1^{\mathrm{T}}N_1\right)<0$$

The above inequality can also be rewritten as

$$\begin{aligned}&\sum_{i=1}^2\sum_{j=1}^2 sym\left\{\Theta_{ij}\otimes\left(\bar{Q}_{ij}A_1\right)\right\}+\\&\left[I_2\otimes\left(\varepsilon N_1^{\mathrm{T}}N_2\right)\right]\left[I_2\otimes\left(2\varepsilon N_2^{\mathrm{T}}N_2\right)\right]^{-1}\left[I_2\otimes\left(\varepsilon N_2^{\mathrm{T}}N_1\right)\right]<0\end{aligned}$$

Invoking Schur complement, the above inequality is equivalent to

$$\left[\begin{array}{cc}\sum\limits_{i=1}^2\sum\limits_{j=1}^2 sym\left\{\Theta_{ij}\otimes\left(\bar{Q}_{ij}A_1\right)\right\}&-I_2\otimes\left(\varepsilon N_1^{\mathrm{T}}N_2\right)\\-I_2\otimes\left(\varepsilon N_2^{\mathrm{T}}N_1\right)&-I_2\otimes\left(2\varepsilon N_2^{\mathrm{T}}N_2\right)\end{array}\right]<0$$

$$\Longleftrightarrow$$

$$sym\left\{\begin{array}{l}\sum\limits_{i=1}^2\sum\limits_{j=1}^2\left\{\Theta_{ij}\otimes\left[\begin{array}{cc}\bar{Q}_{ij}A_1&0\\0&0\end{array}\right]\right\}+\\I_2\otimes\left(\left[\begin{array}{c}0\\I_{n-r}\end{array}\right]\left[\begin{array}{cc}-\varepsilon N_2^{\mathrm{T}}N_1&-\varepsilon N_2^{\mathrm{T}}N_2\end{array}\right]\right)\end{array}\right\}<0$$

$$\Longleftrightarrow$$

$$\begin{aligned}&sym\left\{\begin{array}{l}\sum\limits_{i=1}^2\left\{\Theta_{i1}\otimes\left(\left[\begin{array}{cc}I_r&0\\0&0\end{array}\right]\left[\begin{array}{cc}\bar{Q}_{i1}&0\\0&I_{n-r}\end{array}\right]\left[\begin{array}{cc}A_1&0\\0&I_{n-r}\end{array}\right]\right)\right\}+\\I_2\otimes\left(\left[\begin{array}{c}0\\I_{n-r}\end{array}\right]\left(-\varepsilon N_2^{\mathrm{T}}\right)N\right)\end{array}\right\}+\\&sym\left\{\sum_{i=1}^2\left\{\Theta_{i2}\otimes\left(\left[\begin{array}{cc}I_r&0\\0&0\end{array}\right]\left[\begin{array}{cc}\bar{Q}_{i1}&0\\0&0_{n-r}\end{array}\right]\left[\begin{array}{cc}A_1&0\\0&I_{n-r}\end{array}\right]\right)\right\}\right\}<0\end{aligned}$$

Substituting the decomposed form of triplet (E,A,α) into the above inequality, and let $\tilde{Q}_{i1}=\left[\begin{array}{cc}\bar{Q}_{i1}&0\\0&I_{n-r}\end{array}\right]>0$, $\tilde{Q}_{i2}=\left[\begin{array}{cc}\bar{Q}_{i2}&0\\0&0_{n-r}\end{array}\right]$, $i=1,2$, then

$$sym\left\{\begin{array}{l}\sum\limits_{i=1}^2\sum\limits_{j=1}^2\left\{\Theta_{ij}\otimes\left(N^{\mathrm{T}}E^{\mathrm{T}}M^{\mathrm{T}}\tilde{Q}_{ij}MAN\right)\right\}+\\I_2\otimes\left(N^{\mathrm{T}}(-\varepsilon N_2)M_2AN\right)\end{array}\right\}<0$$

Denote $M^{\mathrm{T}}\tilde{Q}_{ij}M=Q_{ij}$, $M_2^{\mathrm{T}}=E_0$ and $-\varepsilon N_2^{\mathrm{T}}=Q$, where E_0 is a matrix of full column rank. As N is a nonsingular matrix, the above inequality is equivalent to the following:

$$sym\left\{\ \sum_{i=1}^2\sum_{j=1}^2\left\{\Theta_{ij}\otimes\left(E^{\mathrm{T}}Q_{ij}A\right)\right\}+I_2\otimes\left(Q^{\mathrm{T}}E_0^{\mathrm{T}}A\right)\ \right\}<0$$

□

3 Numerical example

In this section, numerical examples are shown to demonstrate the effectiveness of the presented results.

Example 1. Consider a FOS system (5) described with parameters as $\alpha = 0.5$, $E = \begin{bmatrix} 1 & 1 \\ 8 & 8 \end{bmatrix}$ and $A = \begin{bmatrix} -1 & 0 \\ 1 & -9 \end{bmatrix}$.

Firstly, $E_0 = \begin{bmatrix} 8 & -1 \end{bmatrix}^{\mathrm{T}}$ can be chosen to satisfy $E^{\mathrm{T}}E_0 = 0$.

Then, by solving LMI (9) one obtains two real symmetric positive definite matrices

$$Q_{11} = \begin{bmatrix} 2.0349 & -0.8182 \\ -0.8182 & 0.8639 \end{bmatrix}$$

$$Q_{21} = \begin{bmatrix} 0.1167 & 0 \\ 0 & 1.3092 \end{bmatrix}$$

two skew-symmetric matrices

$$Q_{12} = \begin{bmatrix} 0 & 0.8182 \\ -0.8182 & 0 \end{bmatrix}$$

$$Q_{22} = \begin{bmatrix} 0 & 0.8182 \\ -0.8182 & 0 \end{bmatrix}$$

and a matrix $Q = \begin{bmatrix} -2.7139 & -2.7867 \end{bmatrix}$.

Therefore, from Theorem 1, one knows that the fractional-order singular system is admissible.

Example 2. Consider another FOS system (5) with parameters as $\alpha = 0.5$, $A = \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}$ and $E = \begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix}$.

Choose $E_0 = \begin{bmatrix} 3 & -1 \end{bmatrix}^{\mathrm{T}}$ to satisfy $E^{\mathrm{T}}E_0 = 0$.

Then, solving LMI (9) by Matlab, one obtains the following information, which means this FOS is not admissible.

Result: best value of t: 5.404716E−012

f-radius saturation: 21.284 % of $R = 1.00\mathrm{E}{+}009$

Marginal infeasibility: these LMI constraints may be feasible but are not strictly feasible.

4 Conclusion

The issue of admissibility for fractional-order singular system with order belonging to $(0, 1)$ was considered in this paper. The extensions of some basic results of integer order singular system to fractional-order singular system were given, e.g., the definitions of regularity, impulse-free and admissibility; and a sufficient and necessary conditions of admissibility was proposed, which was verified by numerical examples.

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