Semiglobal Stabilization via Output-feedback for a Class of Uncertain Nonlinear Systems

LIU Meng-Liang[1] LIU Yun-Gang[1]

Abstract This paper considers the semiglobal stabilization via output-feedback for a class of uncertain nonlinear systems. Different from the existing results, the systems under investigation possess more serious nonlinearities and unknown control coefficients which substantially increase the difficulty of output-feedback controller design. Combining the backstepping method and output-feedback domination approach, a semiglobal stabilizing controller is explicitly given, which can guarantee that the closed-loop system achieves the semiglobal asymptotic stability under the appropriate choice of design parameter. A simulation example validates the theoretical results and the proposed approach.

Key words Nonlinear systems, unknown control coefficients, semiglobal stabilization, output-feedback

Citation Liu Meng-Liang, Liu Yun-Gang. Semiglobal stabilization via output-feedback for a class of uncertain nonlinear Systems. *Acta Automatica Sinica*, 2013, **39**(12): 2154−2159

DOI 10.3724/SP.J.1004.2013.02154

Stabilization via output-feedback for nonlinear systems has been a major research field in control theory during the last decades, since it needs less information on systems and hence is more practical than that via state-feedback (see e.g., [1−8] and references therein). Mainly because of the theoretical elegance, global stabilization via output-feedback has been paid much attention and plentiful results have been obtained over the past years[1, 4, 7−8]. However, for most nonlinear systems, global stabilization via output-feedback cannot be implied by global stabilization via state feedback plus observability, unlike the case of linear systems. Thus, more severe restrictions are usually imposed on the nonlinear systems, particularly the growth conditions on unmeasurable states[1, 2, 4]. In fact, if the system nonlinearities grow faster than a quadratic nonlinearity with respect to unmeasurable states, there exist counterexamples which cannot achieve global stabilization by any continuous output-feedback[2]. This makes global stabilization impossible in many nonlinear systems which do not satisfy the structural or growth conditions.

Different from global stabilization, semiglobal stabilization means that construction of a stabilizing feedback law yields a region of attraction which contains any a priori given (may be arbitrarily large) compact set[9], and hence has a less ambitious control objective which can meet the needs of practical application. This makes semiglobal stabilization applicable to a much wider class of nonlinear systems than global one[10−15]. The earlier works [11−12] deal with the semiglobal stabilization via output-feedback of fully feedback linearizable nonlinear systems which are generally not globally stabilized via dynamic output-feedback. Reference [13] indicated that semiglobal stabilization via output-feedback can be achieved by uniform observability and global stabilization via state feedback. In some cases, due to presence of mismatched uncertainties and lack of triangularity condition, the nonlinear systems are neither uniformly completely observable nor feedback linearizable. For those systems, semiglobal stabilization was addressed in [10−15]. On the other hand, semiglobal stabilization can relax or remove the severe restrictions needed in the global framework[10, 16]. Mainly because of these, semiglobal stabilization has attracted many attentions.

In this paper, we consider the semiglobal stabilization via output-feedback for a class of uncertain nonlinear systems in the following form:

$$\begin{cases} \dot{\eta}_i = g_i\eta_{i+1} + \psi_i(t,\boldsymbol{\eta},u), \quad i=1,\cdots,n-1 \\ \dot{\eta}_n = g_n u + \psi_n(t,\boldsymbol{\eta},u) \\ y = \eta_1 \end{cases} \tag{1}$$

where $\boldsymbol{\eta} = [\eta_1,\cdots,\eta_n]^{\rm T} \in \mathbf{R}^n$ is the system state with the initial value $\boldsymbol{\eta}(t_0) = \boldsymbol{\eta}_0$; $u \in \mathbf{R}$ and $y \in \mathbf{R}$ are the input and output, respectively; $g_i \neq 0, i=1,\cdots,n$ are unknown constants, called the control coefficients; functions $\psi_i : [t_0,\infty) \times \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}$, $i=1,\cdots,n$ are (piecewise) continuous in the first argument and locally Lipschtiz in the rest two arguments. In what follows, we suppose that only the output y is measurable and available for feedback.

To solve the problem, we make the following assumptions on system (1):

Assumption 1. For $i=1,\cdots,n$,

$$|\psi_i(t,\boldsymbol{\eta},u)| \leq \mu_{\psi_i}(y)\sum_{j=1}^{i}|\eta_j| + \nu_{\psi_i}(y)\sum_{j=2}^{i-1}|\eta_j|^{\frac{i-1}{j-1}} \tag{2}$$

where $\mu_{\psi_i}(\cdot)$ and $\nu_{\psi_i}(\cdot)$ are known nonnegative smooth functions and $\nu_{\psi_1}(\cdot) = \nu_{\psi_2}(\cdot) = 0$.

Assumption 2. The signs of $g_i, i=1,\cdots,n$ are known and there exist known positive constants g_N and g_M, such that

$$g_N \leq |g_i| \leq g_M, \quad i=1,\cdots,n$$

In contrast to [7−8, 17], the systems under investigation contain more serious nonlinearities which permit the higher-order growing unmeasurable states. Compared with [10], the systems considered here have unknown control coefficients which render the output-feedback design more difficult to carry out. In this paper, we combine with backstepping method and output-feedback domination approach to construct a semiglobal stabilizing controller with appropriately adjusted gain, such that the closed-loop trajectories starting from the given region of the initial condition converge to the origin asymptotically.

1 Semiglobal output-feedback controller design

To effectively deal with the unknown control coefficients, we introduce the following coordinates transformation:

$$x_i = \frac{\eta_i}{\prod\limits_{j=i}^{n} g_j}, \qquad i=1,\cdots,n \tag{3}$$

which changes system (1) into the following:

$$\begin{cases} \dot{x}_i = x_{i+1} + \phi_i(t,\boldsymbol{x},u) \\ \dot{x}_n = u + \phi_n(t,\boldsymbol{x},u) \\ y = gx_1 \end{cases} \tag{4}$$

Manuscript received September 13, 2012; accepted March 27, 2013
Supported by National Natural Science Foundations of China (61325016, 61273084, 61233014), Natural Science Foundation for Distinguished Young Scholar of Shandong Province of China (JQ200919), and the Independent Innovation Foundation of Shandong University (2012JC014)
Recommended by Associate Editor CONG Shuang
1. School of Control Science and Engineering, Shandong University, Jinan 250061, China

where $\boldsymbol{x} = [x_1, \cdots, x_n]^{\mathrm{T}}$, $g = \prod_{j=1}^{n} g_j$ and $\phi_i(t, \boldsymbol{x}, u) = \frac{1}{\prod_{j=i}^{n} g_j}\psi_i(t, \boldsymbol{\eta}, u)$, $i = 1, \cdots, n$.

It is necessary to stress that the semiglobal stabilization of system (1) is implied by that of system (4), and therefore we turn to investigate system (4) in the sequel. About the transformed system (4), we have the following proposition to show the growth of its nonlinearities.

Proposition 1. For each nonlinearity ϕ_i of system (4), there holds

$$|\phi_i(t, \boldsymbol{x}, u)| \leq \mu_i(y)\sum_{j=1}^{i}|x_j| + \nu_i(y)\sum_{j=2}^{i-1}|x_j|^{\frac{i-1}{j-1}} \tag{5}$$

where $\mu_i(y) = \frac{\max\{g_M^{n-i+1}, g_M^n\}}{g_N^{n-i+1}}\mu_{\psi_i}(y)$, $\nu_1(y) = 0$, $\nu_2(y) = 0$, $\nu_i(y) = \frac{\max\{g_M^{(n-2)(i-1)}, g_M^{n-i+1}\}}{g_N^{n-i+1}}\nu_{\psi_i}(y)$, $i = 3, \cdots, n$.

Proof. Substituting (3) into (2) and using Assumption 2 directly conclude (5). □

1.1 Observer design

In this paper, motivated by [8, 17], we introduce an input-driven high-gain observer as follows:

$$\begin{cases}\dot{\hat{x}}_i = \hat{x}_{i+1} - L^i a_i \hat{x}_1, i = 1, \cdots, n-1\\ \dot{\hat{x}}_n = u - L^n a_n \hat{x}_1\end{cases} \tag{6}$$

where $L > 1$ is a design parameter to be determined later, and $a_i > 0, i = 1, \cdots, n$ are chosen as the coefficients of Hurwitz polynomial $p(s) = s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n$.

Remark 1. Since g is an unknown coefficient, all the states for system (4) are not measured. Therefore, the full-order observer tuned by the estimate error $x_1 - \hat{x}_1$, as illustrated in [10], is not feasible and cannot be adopted to system (4). Moreover, due to the unknowns in g and $\psi_i(\cdot)$, $i = 1, \cdots, n-1$, the nonlinear terms $\phi_i(\cdot)$ in system (4) are unknown. Hence, some uncertain term (i.e., $\Phi(\cdot)$) arises in the error dynamics system (7) which prevents convergence of high gain observer and makes it difficult to be dealt with by the backstepping method. Motivated by the existing literature, the output-feedback domination approach is used to deal with the uncertain term and guarantees the convergence of the high gain observer.

Let $z_i = \frac{\hat{x}_i}{L^{i-1}}$, $\varepsilon_i = \frac{x_i - \hat{x}_i}{L^{i-1}}$, $i = 1, \cdots, n$, and denote $\boldsymbol{z} = [z_1, \cdots, z_n]^{\mathrm{T}}$, $\boldsymbol{\varepsilon} = [\varepsilon_1, \cdots, \varepsilon_n]^{\mathrm{T}}$. Besides, for notational simplicity, let $z_{n+1} = \frac{u}{L^n}$. Then, by (4) and (6), we have:

$$\dot{\boldsymbol{\varepsilon}} = LA\boldsymbol{\varepsilon} + L\boldsymbol{a}x_1 + \boldsymbol{\Phi}(t, \boldsymbol{x}, u) \tag{7}$$

where $\boldsymbol{\Phi}(t, \boldsymbol{x}, u) = [\phi_1(t, \boldsymbol{x}, u), \frac{\phi_2(t, \boldsymbol{x}, u)}{L}, \cdots, \frac{\phi_n(t, \boldsymbol{x}, u)}{L^{n-1}}]^{\mathrm{T}}$, $\boldsymbol{a} = [a_1, \cdots, a_n]^{\mathrm{T}}$, and

$$A = \begin{bmatrix} -a_1 & 1 & \cdots & 0\\ \vdots & \vdots & \ddots & 0\\ -a_{n-1} & 0 & \cdots & 1\\ -a_n & 0 & \cdots & 0\end{bmatrix}$$

It is easy to see that A is a Hurwitz matrix, for which, there is a symmetric positive-define matrix $P \in \mathbf{R}^{n\times n}$ such that $A^{\mathrm{T}}P + PA \leq -I$, where $I \in \mathbf{R}^{n\times n}$ is identity matrix.

According to (5) and the fact $L > 1$, it can be shown that

$$\frac{\phi_i(t, \boldsymbol{x}, u)}{L^{i-1}} \leq \phi_{1i}(t, \boldsymbol{x}, u) + \phi_{2i}(t, \boldsymbol{x}, u), \quad i = 1, \cdots, n$$

where $\phi_{1i} = \mu_i(y)\left(\left|\frac{1}{g_N^n}y\right| + \sum_{j=2}^{i}(|\varepsilon_j| + |z_j|)\right)$, and $\phi_{21} = \phi_{22} = 0$, $\phi_{2i} = \nu_i(y)\sum_{j=2}^{i-1}\left(|z_j + \varepsilon_j|^{\frac{i-1}{j-1}}\right)$. This is the same as that in [10], and is essential to establish the semiglobal stabilization. For the convenience of later use, let $\boldsymbol{\Phi}_1(t, \boldsymbol{x}, u) = [\phi_{11}, \cdots, \phi_{1n}]^{\mathrm{T}}$ and $\boldsymbol{\Phi}_2(t, \boldsymbol{x}, u) = [\phi_{21}, \cdots, \phi_{2n}]^{\mathrm{T}}$.

Proposition 2. For the error system (7), define $V_{\boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon}) = \boldsymbol{\varepsilon}^{\mathrm{T}}P\boldsymbol{\varepsilon}$. Then there holds

$$\dot{V}_{\boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon}) \leq -\left(\frac{L}{2} - \gamma(y)\right)\|\boldsymbol{\varepsilon}\|^2 + \left(\frac{1}{2} + \frac{2\|P\boldsymbol{a}\|^2 L}{g_N^{2n}}\right)y^2 + \frac{1}{2}\sum_{i=2}^{n} z_i^2 + 2\|\boldsymbol{\varepsilon}^{\mathrm{T}}P\|\cdot\|\boldsymbol{\Phi}_2\| \tag{8}$$

Proof. Clearly, along the solutions of (7), the time-derivative of $V_{\boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon})$ satisfies

$$\dot{V}_{\boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon}) = -L\|\boldsymbol{\varepsilon}\|^2 + 2\boldsymbol{\varepsilon}^{\mathrm{T}}P\boldsymbol{\Phi} + 2L\boldsymbol{\varepsilon}^{\mathrm{T}}P\boldsymbol{a}x_1 \tag{9}$$

Observing that

$$\|\boldsymbol{\Phi}_1(t, \boldsymbol{x}, u)\| \leq \|\boldsymbol{\Phi}_1(t, \boldsymbol{x}, u)\|_1 \leq \frac{\gamma_1(y)}{g_N^n}|y| + \sqrt{n}\gamma_1(y)\|\boldsymbol{\varepsilon}\| + \gamma_1(y)\sum_{i=2}^{n}|z_i|$$

where $\|\boldsymbol{\Phi}_1\|_1 = |\phi_{11}| + \cdots + |\phi_{1n}|$, we have

$$2\boldsymbol{\varepsilon}^{\mathrm{T}}P\boldsymbol{\Phi}_1 \leq \gamma(y)\|\boldsymbol{\varepsilon}\|^2 + \frac{1}{2}y^2 + \frac{1}{2}\sum_{i=2}^{n} z_i^2 \tag{10}$$

where $\gamma_1(y) = \sum_{i=1}^{n}\mu_i(y)$ and $\gamma(y) = \frac{2\|P\|^2}{g_N^{2n}}\gamma_1^2(y) + 2\sqrt{n}\|P\|\gamma_1(y) + 2(n-1)\|P\|^2\gamma_1^2(y)$ are smooth nonnegative functions.

Moreover, there holds

$$2L\boldsymbol{\varepsilon}^{\mathrm{T}}P\boldsymbol{a}x_1 \leq \frac{2}{g_N^n}L\boldsymbol{\varepsilon}^{\mathrm{T}}P\boldsymbol{a}y \leq \frac{L}{2}\|\boldsymbol{\varepsilon}\|^2 + \frac{2L\|P\boldsymbol{a}\|^2}{g_N^{2n}}y^2 \tag{11}$$

Therefore, substituting (10) and (11) into (9) immediately yields (8). □

1.2 Output-feedback controller design

This subsection is devoted to the recursive design steps for a semiglobal output-feedback controller for system (4).

Step 1. Choose $V_1(\boldsymbol{\varepsilon}, \xi_1) = V_{\boldsymbol{\varepsilon}} + \frac{1}{2}y^2$, where $\xi_1 = y$. By (8), a direct calculation gives

$$\dot{V}_1 \leq -\left(\frac{L}{2} - \gamma(y)\right)\|\boldsymbol{\varepsilon}\|^2 + \left(\frac{1}{2} + \frac{2\|P\boldsymbol{a}\|^2 L}{g_N^{2n}}\right)y^2 + \frac{1}{2}\sum_{i=2}^{n} z_i^2 + y\left(gL\varepsilon_2 + gLz_2 + g\phi_1\right) + 2\|\boldsymbol{\varepsilon}^{\mathrm{T}}P\|\cdot\|\boldsymbol{\Phi}_2\| \tag{12}$$

By the method of completing squares, we have:

$$\begin{cases} Lgy\varepsilon_2 \leq L|gy\varepsilon_2| \leq \frac{L}{4}\|\boldsymbol{\varepsilon}\|^2 + Lg_M^{2n}y^2\\ gy\phi_1 \leq \frac{g_M^n\mu_1(y)}{g_N^n}y^2\end{cases}$$

Substituting this into (12) concludes

$$\dot V_1 \le -\left(\frac{L}{4}-\gamma(y)\right)\|\boldsymbol\varepsilon\|^2+\left(\frac12+\frac{g_M^n\mu_1(y)}{g_N^n}\right)y^2+ L\left(\frac{2\|P\boldsymbol a\|^2}{g_N^{2n}}+g_M^{2n}\right)y^2+\frac12\sum_{i=2}^n z_i^2+ gLyz_2+2\|\boldsymbol\varepsilon^{\mathrm T}P\|\cdot\|\boldsymbol\Phi_2\| \tag{13}$$

Choose the virtual controller $\alpha_1(\cdot)$ as follows:

$$\alpha_1=-c_1y \tag{14}$$

where $c_1=\left(\frac{2\|P\boldsymbol a\|^2}{g_N^{2n}}+g_M^{2n}+1\right)\frac{\mathrm{sgn}(g)}{g_N^n}$. Let $\xi_2=z_2-\alpha_1$. Then noting $\frac12 z_2^2\le\xi_2^2+\alpha_1^2$, and substituting (14) into (13) yield

$$\dot V_1\le-\left(\frac L4-\gamma(y)\right)\|\boldsymbol\varepsilon\|^2-(L-\rho_1(y))\,y^2+ \frac12\sum_{i=3}^n z_i^2+\xi_2^2+Lgy\xi_2+2\|\boldsymbol\varepsilon^{\mathrm T}P\|\cdot\|\boldsymbol\Phi_2\| \tag{15}$$

where $\rho_1(y)=\frac12+\frac{g_M^n\mu_1(y)}{g_N^n}+c_1^2$.

Step 2. Choose $V_2(\boldsymbol\varepsilon,\boldsymbol\xi_{[2]})=V_1+\frac12\xi_2^2$ for this step, where in the sequel $\boldsymbol\xi_{[i]}=[\xi_1,\cdots,\xi_i]^{\mathrm T}$, $i=2,\cdots,n$ and particularly $\boldsymbol\xi=\boldsymbol\xi_{[n]}$. In view of (6) and (14), we obtain

$$\dot\xi_2=Lz_3-La_2\hat x_1+c_1\left(Lg\varepsilon_2+Lgz_2+g\phi_1\right)$$

Then by (15), the time-derivative of V_2 along the solutions of (4) and (6) satisfies

$$\dot V_2=-\left(\frac L4-\gamma(y)\right)\|\boldsymbol\varepsilon\|^2-(L-\rho_1(y))\,y^2+ \frac12\sum_{i=3}^n z_i^2+\xi_2^2+Lgy\xi_2+2\|\boldsymbol\varepsilon^{\mathrm T}P\|\cdot\|\boldsymbol\Phi_2\|+ \xi_2\left(Lz_3-La_2\hat x_1+c_1\left(Lg\varepsilon_2+Lgz_2+g\phi_1\right)\right) \tag{16}$$

By using the method of completing squares, we obtain the following estimations for the terms on the right-hand side of (16):

$$\begin{cases} Lgy\xi_2\le\frac{3Lg_M^{2n}}{2}\xi_2^2+\frac L6y^2\\ -La_2\xi_2\hat x_1\le\frac L6y^2+\frac{3La_2^2}{2g_N^{2n}}\xi_2^2+\frac L{16}\|\boldsymbol\varepsilon\|^2+4La_2^2\xi_2^2\\ Lc_1g\xi_2\varepsilon_2\le\frac L{16}\|\boldsymbol\varepsilon\|^2+4Lc_1^2g_M^{2n}\xi_2^2\\ Lc_1g\xi_2z_2\le L|c_1|g_M^n\xi_2^2+\frac{3Lc_1^4g_M^{2n}}{2}\xi_2^2+\frac L6y^2\\ c_1g\xi_2\phi_1\le\frac{g_M^{2n}c_1^2\mu_1^2(y)}{2}y^2+\frac12\xi_2^2 \end{cases}$$

Substituting this into (16) leads to

$$\dot V_2\le-\left(\frac L8-\gamma(y)\right)\|\boldsymbol\varepsilon\|^2+\frac12\sum_{i=3}^n z_i^2+L\xi_2z_3+ (c_2-1)L\xi_2^2-\left(\frac L2-\rho_1(y)-\frac{g_M^{2n}c_1^2\mu_1^2(y)}{2}\right)y^2+ \frac32\xi_2^2+2\|\boldsymbol\varepsilon^{\mathrm T}P\|\cdot\|\boldsymbol\Phi_2\|$$

where $c_2=1+4a_2^2+\frac{3g_M^{2n}}{2}+\frac{3a_2^2}{2g_N^{2n}}+4c_1^2g_M^{2n}+|c_1|g_M^n+\frac{3c_1^4g_M^{2n}}{2}$.

Choose the virtual controller α_2 as follows:

$$\alpha_2=-c_2\xi_2 \tag{17}$$

Then, by defining $\xi_3=z_3-\alpha_2$ and letting $\rho_2(y)=\rho_1(y)+\frac{c_1^2g_M^{2n}\mu_1^2(y)}{2}$, we have:

$$\dot V_2\le-\left(\frac L8-\gamma(y)\right)\|\boldsymbol\varepsilon\|^2-\left(\frac L2-\rho_2(y)\right)y^2- \left(L-\frac32-c_2^2\right)\xi_2^2+\frac12\sum_{i=4}^n z_i^2+\xi_3^2+ L\xi_2\xi_3+2\|\boldsymbol\varepsilon^{\mathrm T}P\|\cdot\|\boldsymbol\Phi_2\|$$

Step $\boldsymbol{k\,(k=3,\cdots,n)}$. Suppose that the previous $k-1$ steps had completed and yielded smooth virtual controllers $\alpha_i(\xi_i)$, $i=1,\cdots,k-1$ defined by $\alpha_1=-c_1y$, $\alpha_i=-c_i\xi_i$, $\xi_i=z_i-\alpha_{i-1}, i=2,\cdots,k-1$ with c_i's being non-zero constants, such that

$$\dot V_{k-1}\le-\left(\frac L{2^k}-\gamma(y)\right)\|\boldsymbol\varepsilon\|^2-\left(\frac L{2^{k-2}}-\rho_{k-1}(y)\right)y^2- \sum_{i=2}^{k-1}\left(\frac L{2^{k-1-i}}-\frac32-c_i^2\right)\xi_i^2+\frac12\sum_{i=k+1}^n z_i^2+ \xi_k^2+L\xi_k\xi_{k-1}+2\|\boldsymbol\varepsilon^{\mathrm T}P\|\cdot\|\boldsymbol\Phi_2\|$$

where $V_{k-1}=V_1+\frac12\sum_{j=2}^{k-1}\xi_j^2$, and ρ_{k-1} is a positive function of y.

Now, choose the smooth function $V_k(\boldsymbol\varepsilon,\boldsymbol\xi_{[k]})=V_{k-1}+\frac12\xi_k^2$, $\xi_k=z_k-\alpha_{k-1}$ for step k. Observing that

$$\dot\xi_k=Lz_{k+1}+L\bar a_kz_1+Lc_{k-1}z_k+L\sum_{i=2}^{k-2}\Big(\prod_{j=i}^{k-1}c_j\Big)z_{i+1}+ \Big(\prod_{j=1}^{k-1}c_j\Big)\left(Lg\varepsilon_2+Lgz_2+g\phi_1\right)$$

with $\bar a_k=-a_k-\sum_{i=2}^{k-1}\Big(\prod_{j=i}^{k-1}c_j\Big)a_i$, we have:

$$\dot V_k=\dot V_{k-1}+\xi_k\dot\xi_k\le-\left(\frac L{2^k}-\gamma(y)\right)\|\boldsymbol\varepsilon\|^2- \left(\frac L{2^{k-2}}-\rho_{k-1}(y)\right)y^2-\sum_{i=2}^{k-1}\left(\frac L{2^{k-1-i}}-\frac32-c_i^2\right)\xi_i^2+ \frac12\sum_{i=k+1}^n z_i^2+\xi_k^2+L\xi_k\xi_{k-1}+\xi_k\dot\xi_k+2\|\boldsymbol\varepsilon^{\mathrm T}P\|\cdot\|\boldsymbol\Phi_2\| \tag{18}$$

To deduce the function α_k, we obtain the following estimations for $L\xi_k\xi_{k-1}$ and all terms in $\xi_k\dot{\xi}_k$:

$$\begin{cases} L\xi_k\xi_{k-1} \le \frac{3L}{2}\xi_k^2 + \frac{L}{6}\xi_{k-1}^2 \\ c_{k-1}L\xi_k z_k \le c_{k-1}\left(1+\frac{3}{2}c_{k-1}^3\right)L\xi_k^2 + \frac{L}{6}\xi_{k-1}^2 \\ \bar{a}_k L\xi_k z_1 \le \frac{L}{2^k}y^2 + L2^{k-2}\bar{a}_k^2\left(\frac{1}{g_N^{2n}}+4\right)\xi_k^2 + \frac{L}{2^{k+2}}\|\boldsymbol{\varepsilon}\|^2 \\ \Big(\prod\limits_{j=1}^{k-1}c_j\Big)g\xi_k\phi_1 \le \frac{1}{2}\xi_k^2 + \frac{\Big(\prod\limits_{j=1}^{k-1}c_j^2\Big)g_M^{2n}\mu_1^2(y)}{2}y^2 \\ \Big(\prod\limits_{j=1}^{k-1}c_j\Big)Lg\xi_k\varepsilon_2 \le \frac{L}{2^{k+2}}\|\boldsymbol{\varepsilon}\|^2 + 2^k g_M^{2n}L\Big(\prod\limits_{j=1}^{k-1}c_j^2\Big)\xi_k^2 \\ \Big(\prod\limits_{j=1}^{k-1}c_j\Big)Lg\xi_k z_2 \le \\ \qquad 2^{k-2}L\Big(\prod\limits_{j=1}^{k-1}c_j^2\Big)g_M^{2n}\left(\frac{3}{4}+c_1^2\right)\xi_k^2 + \\ \qquad \frac{L}{3\cdot 2^{k-2}}\xi_2^2 + \frac{L}{2^k}y^2 \\ \sum\limits_{i=2}^{k-2}\Big(\prod\limits_{j=i}^{k-1}c_j\Big)L\xi_k z_{i+1} \le \\ \qquad 3L\sum\limits_{i=2}^{k-2}\Big(2^{k-3-i}(1+c_i^2)\prod\limits_{j=i}^{k-1}c_j^2\Big)\xi_k^2 + \\ \qquad \sum\limits_{i=3}^{k-2}\frac{L}{2^{k-i}}\xi_i^2 + \frac{L}{6}\xi_{k-1}^2 + \frac{L}{3\cdot 2^{k-3}}\xi_2^2 \end{cases}$$

Substituting this into (18) concludes

$$\begin{aligned} \dot{V}_k =& \dot{V}_{k-1} + \xi_k\dot{\xi}_k \le \\ & -\left(\frac{L}{2^{k+1}} - \gamma(y)\right)\|\boldsymbol{\varepsilon}\|^2 - \left(\frac{L}{2^{k-1}} - \rho_k(y)\right)y^2 - \\ & \sum_{i=2}^{k-1}\left(\frac{L}{2^{k-i}} - \frac{3}{2} - c_i^2\right)\xi_i^2 + \frac{1}{2}\sum_{i=k+1}^{n}z_i^2 + \frac{3}{2}\xi_k^2 + \\ & L(c_k-1)\xi_k^2 + L\xi_k z_{k+1} + 2\|\boldsymbol{\varepsilon}^{\mathrm{T}}P\|\cdot\|\boldsymbol{\Phi}_2\| \end{aligned}$$

where

$$\begin{cases} \rho_k = \rho_{k-1} + \frac{g_M^{2n}\prod\limits_{j=1}^{k-1}c_j^2}{2}\mu_1^2(y) \\ c_k = \frac{5}{2} + 2^k g_M^{2n}\Big(\prod\limits_{j=1}^{k-1}c_j^2\Big) + \left(2^k + \frac{2^{k-2}}{g_N^{2n}}\right)\bar{a}_k^2 + \\ \qquad 3\sum\limits_{i=2}^{k-2}\left(2^{k-3-i}(1+c_i^2)\Big(\prod\limits_{j=i}^{k-1}c_j^2\Big)\right) + \\ \qquad 2^{k-2}g_M^{2n}\Big(\prod\limits_{j=1}^{k-1}c_j^2\Big)\left(\frac{3}{4}+c_1^2\right) + \\ \qquad c_{k-1}\left(1+\frac{3}{2}c_{k-1}^3\right) \end{cases} \tag{19}$$

By choosing $\alpha_k = -c_k\xi_k$ and letting $\xi_{k+1} = z_{k+1} - \alpha_k$, we have

$$\begin{aligned} \dot{V}_k \le & -\left(\frac{L}{2^{k+1}} - \gamma(y)\right)\|\boldsymbol{\varepsilon}\|^2 - \left(\frac{L}{2^k} - \rho_k(y)\right)y^2 - \\ & \sum_{i=2}^{k}\left(\frac{L}{2^{k-i}} - \frac{3}{2} - c_i^2\right)\xi_i^2 + \frac{1}{2}\sum_{i=k+2}^{n}z_i^2 + \\ & L\xi_k\xi_{k+1} + \xi_{k+1}^2 + 2\|\boldsymbol{\varepsilon}^{\mathrm{T}}P\|\cdot\|\boldsymbol{\Phi}_2\| \end{aligned} \tag{20}$$

By now, we complete the entire design steps.

From the n-th step of the above entire design procedure, we can obtain $\alpha_n(\xi_n)$, and hence the actual controller:

$$u = L^n\alpha_n \tag{21}$$

where α_n is recursively defined as follows:

$$\begin{cases} \alpha_i = -c_i\xi_i,\ i = 1,\cdots,n \\ \xi_i = z_i - \alpha_{i-1},\ i = 2,\cdots,n \end{cases}$$

Noting that $z_{n+1} = \frac{u}{L^n}$ and thus $\xi_{n+1} = 0$, from (20) and (21), it follows that

$$\begin{aligned} \dot{V}_n \le & -\left(\frac{L}{2^{n+1}} - \gamma(y)\right)\|\boldsymbol{\varepsilon}\|^2 - \left(\frac{L}{2^n} - \rho_n(y)\right)y^2 - \\ & \sum_{i=2}^{n}\left(\frac{L}{2^{n-i}} - \frac{3}{2} - c_i^2\right)\xi_i^2 + 2\|\boldsymbol{\varepsilon}^{\mathrm{T}}P\|\cdot\|\boldsymbol{\Phi}_2\| \end{aligned} \tag{22}$$

where ρ_n and c_i's are recursively defined by (19), and $V_n = V_1 + \frac{1}{2}\sum_{j=2}^n\xi_j^2$.

It should be pointed out that the design parameter L is not yet specified, which determines whether or not the closed-loop states asymptotically converge to the origin, for the given region of the initial condition.

2 Main results

We are now in a position to summarize the main result of the paper into the following theorem.

Theorem 1. Consider system (1) under Assumptions 1 and 2. Based on the high-gain observer (6) and the output-feedback controller (21), if the design parameter L is suitably chosen, then the semiglobal stabilization can be achieved.

Proof. We prove the theorem by referring to that in [10]. By the above definitions and choices, we see that $\overline{V}_n(\boldsymbol{\varepsilon},\boldsymbol{z}) := V_n(\boldsymbol{\varepsilon},\boldsymbol{\xi})$ is positive definite and radially unbounded. Define $N_K = \left\{\boldsymbol{\zeta}\,\middle|\,\|\boldsymbol{\zeta}\| \le \frac{K}{2}, \boldsymbol{\zeta}\in\mathbf{R}^n\right\}$, and $\Omega_K = \left\{(\boldsymbol{\varepsilon},\boldsymbol{z})\,\middle|\,V_n \le M_K,\ M_K = \max_{\{\|\boldsymbol{\varepsilon}\|\le K,\,\|\boldsymbol{z}\|\le K\}}V_n\right\}$, where $K > 1$ is arbitrary. Clearly, N_K and Ω_K are nonempty compact sets.

Noting the preceding definitions, particularly $\boldsymbol{\Phi}_2(t,\boldsymbol{x},u)$, ξ_i's and α_i's, for any $(\boldsymbol{\varepsilon},\boldsymbol{z})\in\Omega_K$, there holds

$$\begin{aligned} & \|\boldsymbol{\varepsilon}^{\mathrm{T}}P\|\cdot\|\boldsymbol{\Phi}_2\| \le \|\boldsymbol{\varepsilon}^{\mathrm{T}}P\|\cdot\|\boldsymbol{\Phi}_2\|_1 \le \\ & \quad \nu(y)\|\boldsymbol{\varepsilon}^{\mathrm{T}}P\|\sum_{i=3}^{n}\sum_{j=2}^{i-1}\left(|z_j+\varepsilon_j|^{\frac{i-1}{j-1}}\right) = \\ & \quad \nu(y)\|\boldsymbol{\varepsilon}^{\mathrm{T}}P\|\sum_{i=3}^{n}\sum_{j=2}^{i-1}\left(|\xi_j - c_{j-1}\xi_{j-1}+\varepsilon_j|^{\frac{i-1}{j-1}}\right) \le \\ & \quad \lambda_1\left(\|\boldsymbol{\xi}\|^2 + \|\boldsymbol{\varepsilon}\|^2\right) \end{aligned} \tag{23}$$

where $\nu(y) = \sum_{i=1}^n\nu_i(y)$, $\|\boldsymbol{\Phi}_2\|_1 = |\phi_{21}| + \cdots + |\phi_{2n}|$ and λ_1 is a positive constant which depends on Ω_K and is independent of L. In deriving the above inequality, the boundedness of z_i's and ε_i's (as well as ξ_i's and ε_i's) on compact set Ω_K has been used.

Hence, noting that $y = z_1 + \varepsilon_1$, and by the continuity of $\gamma(y)$ and $\rho_i(y)'$s, there exists a positive constant λ_2, such that on Ω_K, there hold $\gamma(y) \le \lambda_2$ and $\rho_n(y) \le \lambda_2$.

By choosing

$$\begin{aligned} L \ge \max\Big\{ & 2^{n+1}(2\lambda_1+\lambda_2),\quad 2^{n-i-1}(3+2c_i^2+2\lambda_1), \\ & i = 2,\cdots,n\Big\} \end{aligned} \tag{24}$$

the right hand side of (22) becomes negative definite, so$\dot{\overline{V}}_n(\boldsymbol{\varepsilon},\boldsymbol{z})$ will be negative definite. In this case, the trajectories of the closed-loop system starting from Ω_K will stay in this compact set forever. Moreover,

$$\lim_{t\to+\infty}\boldsymbol{\varepsilon}(t)=0,\quad \lim_{t\to+\infty}\boldsymbol{z}(t)=0$$

from the definitions of ε_i's and z_i's, we have

$$\hat{x}_i=L^{i-1}z_i,\quad x_i=\hat{x}_i+L^{i-1}\varepsilon_i$$

when ε_i and z_i converge to zero, we can easily get x_i and $\hat{x}_i$ converge to zero. Hence the set Ω_K is the domain of attraction of the closed-loop system.

Noting the previous definitions of ε_i's and z_i's, we have the following relation for any arbitrarily large K,

$$\boldsymbol{x}\in N_K,\hat{\boldsymbol{x}}\in N_K\Rightarrow\|\boldsymbol{\varepsilon}\|\le K,\|\boldsymbol{z}\|\le K\Rightarrow(\boldsymbol{\varepsilon},\boldsymbol{z})\in\Omega_K$$

Consequently, we have the conclusion that starting from any points in $N_K\times N_K$, the trajectory will stay in the compact set Ω_K and tend to the origin. □

Remark 2. In this paper, we use the method of completing squares rather than the weighted mean square inequality (i.e., $2d_1d_2\le\frac{d_1^2}{\mu}+\mu d_2^2$ with a weighted coefficient $\mu>0$) to establish some estimates in the construction of the desirable controller, since the new parameter μ will make the controller design and stability analysis much complicated. It is necessary to point out that, the introduction of weighted coefficient μ may affect the magnitude of the design parameter L, but how to choose the appropriate μ to make L smaller is difficult. In fact, the determination of L depends on the choosing of key parameters λ_1, λ_2 and c_i, $i=1,2,\cdots,n$ which have complex nonlinear relationship. Once weighted mean square inequality is used, λ_1, λ_2 and c_i, $i=1,2,\cdots,n$ would depend on μ. Therefore, how to choose the appropriate μ so as to specify the appropriate λ_1, λ_2 and c_i, $i=1,2,\cdots,n$ and hence to make L smaller would be much difficult to achieve.

Remark 3. It is clear that L is determined provided that the parameters λ_1, λ_2 and $c_i, i=2,\cdots,n$ are given. Obviously, $c_i, i=2,\cdots,n$ are directly given through the construction of the virtual controller α_i's (for example, see (19)). In fact, the purpose of introducing λ_1 and λ_2 is to facilitate the choosing of an appropriate L which can finally guarantee the stability of closed-loop system. For details, λ_1 is the estimation gain of $\|\boldsymbol{\varepsilon}^{\rm T}P\|\cdot\|\boldsymbol{\Phi}_2\|$ in the domain Ω_K, and λ_2 is the common upper bound of $\gamma(y)$ and $\rho_n(y)$ in the domain Ω_K. Therefore, we can choose λ_1 by estimating $\|\boldsymbol{\varepsilon}^{\rm T}P\|\cdot\|\boldsymbol{\Phi}_2\|$ which is formulated as in (23), and choose $\lambda_2=\max\{\sup\gamma(y),\sup\rho_n(y)\}$ in the compact set of y. On the other hand, we can specify an appropriate design parameter L from (24) by conservatively choosing λ_1 and λ_2. Specifically, λ_1 can be chosen based on the estimations of $\|\boldsymbol{\varepsilon}^{\rm T}P\|\cdot\|\boldsymbol{\Phi}_2\|$ and $\|\boldsymbol{\xi}\|^2+\|\boldsymbol{\varepsilon}\|^2$ on the given compact set Ω_K, and λ_2 can be chosen based on the estimations of $\gamma(y)$ and $\rho_n(y)$ in the domain Ω_K.

3 Simulation example

In this section, a numerical example is provided to illustrate the correctness and effectiveness of the theoretical results by considering the following third-order nonlinear system:

$$\begin{cases}\dot{x}_1=g_1x_2+2x_1\sin(x_1)\\ \dot{x}_2=g_2x_3+x_1x_2\\ \dot{x}_3=g_3u+x_1(x_1+x_2^2\sin(x_2))\\ y=x_1\end{cases}$$

where $0.5\le g_i\le 1.5, i=1,2,3$. It is easy to verify that the system satisfies Assumptions 1 and 2.

Choose $a_1=\frac{5}{4}, a_2=\frac{1}{2}, a_3=\frac{1}{16}$ such that A is Hurwitz and design the observer as follows:

$$\begin{cases}\dot{\hat{x}}_1=\hat{x}_2-La_1\hat{x}_1\\ \dot{\hat{x}}_2=\hat{x}_3-L^2a_2\hat{x}_1\\ \dot{\hat{x}}_3=u-L^3a_3\hat{x}_1\end{cases}$$

According to Section 2, we can obtain a semiglobal stabilizing output-feedback controller in the form as (21). Then we can select $L=18$, the initial conditions are $\boldsymbol{x}(0)=[0,0.5,-0.5]^{\rm T}$, $\hat{\boldsymbol{x}}(0)=[0.1,0,-0.4]^{\rm T}$. Figs. 1 ~ 4 depict the simulation results. From the figures, we see that the output-feedback stabilizer can guarantee that the trajectories starting from the given initial domination indeed converge to the origin asymptotically.

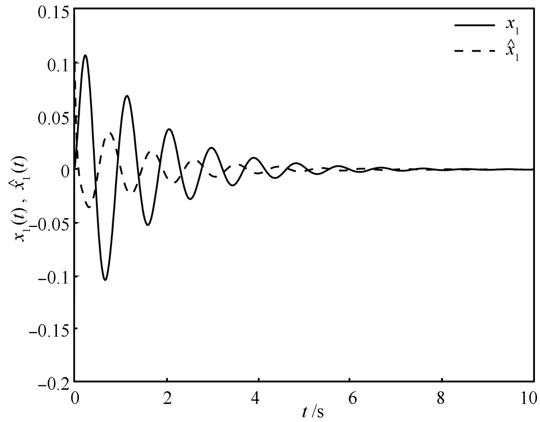

Fig. 1 State x_1 and its estimation $\hat{x}_1$

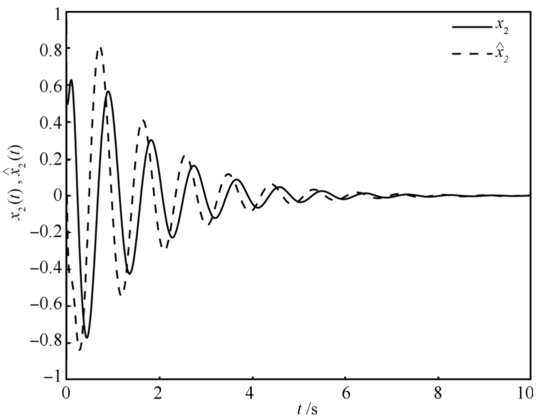

Fig. 2 State x_2 and its estimation $\hat{x}_2$

4 Concluding remarks

Compared with global stabilization, semiglobal stabilization has less restriction and much more practical applications. This paper has addressed the semiglobal stabilization via output-feedback for a class of nonlinear systems with unknown control coefficients. The semiglobal stable controller via output-feedback has been constructed by using backstepping method and output-feedback domination approach. Under the appropriate choice of gain, the designed controller guarantees the closed-loop system semiglobally asymptotically stable. It is worth mentioning that the semiglobal output-feedback stabilization of high-order nonlinear systems with unknown control coefficients is far more complicated and difficult than the special case considered in present paper. This problem is currently under our study. Moreover, when the design parameter L is large, the feedback gain will quickly become unacceptable in practice, so how to design a controller with small feedback gain is also meaningful and deserves further investigation.

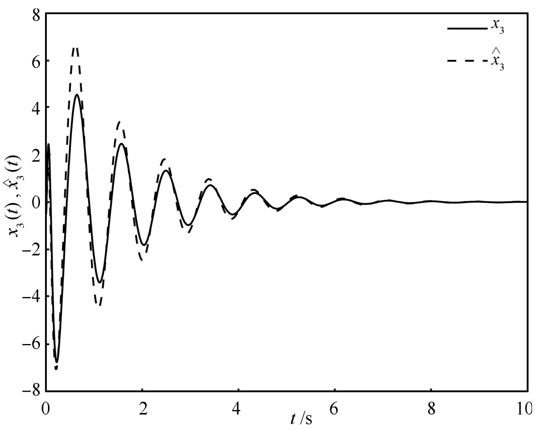

Fig. 3 State x_3 and its estimation $\hat{x}_3$

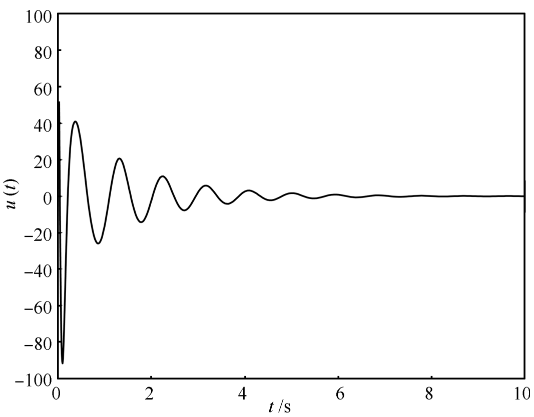

Fig. 4 Controller u

References

1 Tsinias J. A theorem on global stabilization of nonlinear systems by linear feedback. *Systems and Control Letters*, 1991, **17**(5): 357−362

2 Mazenc F, Praly L, Dayawansa W P. Global stabilization by output feedback: examples and counterexamples. *Systems and Control Letters*, 1994, **23**(2): 119−125

3 Teel A, Praly L. Tools for semiglobal stabilization by partial state and output feedback. *SIAM Journal on Control and Optimization*, 1995, **33**(5): 1443−1488

4 Qian C J, Lin W. Output feedback control of a class of nonlinear systems: a nonseparation principle paradigm. *IEEE Transactions on Automatic Control*, 2002, **47**(10): 1710−1715

5 Khalil H K. *Nonlinear Systems (Third edition)*. New Jersey: Prentice Hall, 2002

6 Isidori A. A tool for semi-global stabilization of uncertain non-minimum-phase nonlinear systems via output feedback. *IEEE Transactions on Automatic Control*, 2000, **45**(10): 1817−1827

7 Liu Y G. Global stabilization by output feedback for a class of nonlinear systems with uncertain control coefficients and unmeasured states dependent growth. *Science in China, Series F: Information Sciences*, 2008, **51**(10): 1508−1520

8 Jiang Z P, Mareels I, Hill D J, Huang J. A unifying framework for global regulation via nonlinear output feedback: from ISS to iISS. *IEEE Transactions on Automatic Control*, 2004, **49**(4): 549−562

9 Saberi A, Lin Z L, Teel A R. Control of linear systems with saturating actuators. *IEEE Transactions on Automatic Control*, 1996, **41**(3): 368−378

10 Qian C J. Semi-global stabilization of a class of uncertain nonlinear systems by linear output feedback. *IEEE Transactions on Circuits and Systems*, 2005, **52**(4): 218−222

11 Esfandiari F, Khalil H K. Output feedback stabilization of fully linearizable systems. *International Journal of Control*, 1992, **56**(5): 1007−1037

12 Khalil H K, Esfandiari F. Semiglobal stabilization of a class of nonlinear systems using output feedback. *IEEE Transactions on Automatic Control*, 1993, **38**(9): 1412−1415

13 Teel A, Praly L. Global stabilizability and observability imply semi-global stabilizability by output feedback. *Systems and Control Letters*, 1994, **22**(5): 313−325

14 Battilotti S. Semiglobal stabilization via measurement feedback for systems in triangular form. In: Proceedings of the 38th IEEE Conference on Decision and Control. Phoenix, USA: IEEE, 1999. 837−841

15 Lin W, Qian C J. Semi-global robust stabilization of nonlinear systems by partial state and output feedback. In: Proceedings of the 37th IEEE Conference on Decision and Control. Tampa, USA: IEEE, 1998. 3105−3110

16 Yang B, Lin W. Semi-global output feedback stabilization of non-uniformly observable and non-smoothly stabilizable systems. In: Proceedings of the 44th IEEE conference on Decision and Control and European Control Conference. Seville, Spain: IEEE, 2005. 4207−4212

17 Wei C L, Wang Q D. Global output feedback asymptotic stabilization for a class of nonlinear systems. In: Proceedings of the 29th Chinese Control Conference. Beijing, China: IEEE, 2010. 369−374

LIU Meng-Liang Ph. D. candidate at the School of Control Science and Engineering, Shandong University. His research interest covers nonlinear output feedback control and adaptive control theory. E-mail: mengliang804@163.com

LIU Yun-Gang Professor at the School of Control Science and Engineering, Shandong University. His research interest covers adaptive control, stochastic systems, optimal control, and observer design. Corresponding author of this paper. E-mail: lygfr@sdu.edu.cn