Nonlinear Control for Multi-agent Formations with Delays in Noisy Environments

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Abstract In this paper, we investigate the nonlinear control problem for multi-agent formations with communication delays in noisy environments and in directed interconnection topologies. A stable theory of stochastic delay differential equations is established and then some sufficient conditions are obtained based on this theory, which allow the required formations to be gained at exponentially converging speeds with probability one for time-invariant formations, time-varying formations, and time-varying formations for trajectory tracking under a special "multiple leaders" framework. Some numerical simulations are also given to illustrate the effectiveness of the theoretical results.

Key words Formation control, multi-agent systems, delays, noise

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Multi-agent systems formation control is a consensus-related problem (first-order dynamics[1−4] or second-order dynamics[5−8], to name a few) that has been intensively investigated across many scientific disciplines in the past few years with a lot of results applicable to a range of engineering control problems. Rapid advances have been made, for example, in the cooperative control of some notoriously difficult systems such as unmanned aerial vehicles (UAVS), autonomous underwater vehicles (AUVS) and mobile robot systems (MRS), where there is a great need for consensus of both the agent states (such as positions, velocities, etc.) and the group formations (i.e. the alignments).

Many formation control methods have been proposed in [9−18]. For example, some methods that use concepts from graph theory and dynamical systems were proposed in [13−14]. In [9, 12], the authors discussed the formation control problems for a class of fractional-order dynamics by using the distributed communication protocols. From the viewpoint of network connectivity, the authors of [11, 14] designed some decentralized formation controllers. In [16] and [17], the authors investigated the formation control problems under the conditions of noise disturbance and communication delays, respectively. Particularly, a kind of formation control framework was proposed in [12, 19−21], which classifies all information into global and local classes with the global information being accessible to only a few of the selected agents and the local information to all. This control protocol is very practical in design and highly suitable for applications, though coupling time delays and transmission noise perturbations are completely ignored.

Time delays, however, are always present in real-world physical systems[21−24] because of the unavoidable time-lags in the exchange of information between system nodes (time-lags could be caused, for example, by the finite switching speeds of the amplifiers in an electrical circuit) and time-delay-free protocols are prone to be unstable in numerical simulation experiments. By the same token, noise disturbances are prevalent in nature with system properties (such as agent group motions) being typically susceptible to their effects[25−27]. Despite the knowledge that we have gained in noise-free formation problems over the years, however, general results on time-delayed noise-perturbed complex systems are still very much a scarcity.

In this paper, based on our previous works of [18, 28−29], we further investigate a class of nonlinear control problems for multi-agent formations with single-integrator dynamic that possess time-delayed couplings and directed graphs under noisy environments. In particular, we apply a special "multiple leaders" framework and classify the agents into leaders and followers with the global information being accessible to only the leaders. We allow a small number of the leaders (called the leaders of the leaders) to regulate their states in accordance with their state deviations and to pin the other leaders into attaining the expected formations and allow a small number of followers (called the leaders of the followers) to do the same with other followers. It is also assumed that the followers′ states are being continuously updated with the latest global formation information during the formation process and that the leaders′ own dynamics remain perfectly unaffected by those of their followers. Our aim is to obtain sufficient conditions for attaining time-invariant formations, time-varying formations and time-varying formations for trajectory tracking and extend the results of [12] to the case of communication-delayed and noise-disturbed nonlinear multi-agent dynamics.

This paper has three main contributions. Firstly, a stability theory of stochastic delay differential equations has been established, which can be used to resolve a large class of nonlinear control problems. Secondly, our treatment of multiple-agent classes provides a good general model for commercial corporate structures in which multiple level management systems are prevalent and specific duties are assigned to the leaders at different levels. Thirdly, we consider the effects of coupling delays and noise disturbance on formation control, and the upper bound of the time delays is derived. Such theoretical results can be applied to the actual noise-disturbed multi-agent systems with nonlinear coupling function and time delays running in asymmetric communication topologies.

This paper is organized as follows. In Section 1, we formulate the formation control problems. In Section 2, we discuss the stability theory of stochastic delay differential equations and present three main formation control protocols in Section 3. In Section 4, we consider some examples and show some simulation results, and conclusions are

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drawn in Section 5.

Throughout this paper and unless specified, let $|\cdot|$ be the Euclidean norm. If A is a vector or a matrix, then A^{T} denotes its transpose. If A is a matrix, $\|A\|$ denotes the operator norm of A, i.e., $\|A\| = \sup\{|Ax| : |x| = 1\}$. Moreover, let $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$ be the maximum and minimum eigenvalues of symmetric matrix A, respectively. For any stochastic variable x, let $\mathrm{E}(x)$ be its mathematical expectation.

1 Model description and preliminaries

In this paper, we consider a system that is made up of ℓ leaders (indexed by $1, 2, \cdots, \ell$) and $N-\ell$ followers (indexed by $\ell+1, \ell+2, \cdots, N$). We assume that the motions of the leaders are independent and that the motions of the followers are dependent on those of their leaders and other agents. The inter-relationships between the agents can then be conveniently described by a digraph $\mathcal{G}$ with the N agents at the vertices.

Let $x_i \in \mathbf{R}^n$ be the state of agent i with dynamics

$$\dot{x}_i = u_i \tag{1}$$

where u_i is the state feedback called the protocol to be designed.

In order to facilitate our analysis, this paper only considers the one-dimensional case, i.e., $n = 1$. However, similar analysis can also be done for the higher-dimensional case by means of Kronecker product. Let $N^{\ell} = \{1, 2, \cdots, \ell\}$, $N^f = \{\ell+1, \cdots, N\}$, $\boldsymbol{x} = (x_1, \cdots, x_N)^{\mathrm{T}}$ and $\boldsymbol{x}^{\ell} = (x_1, \cdots, x_{\ell})^{\mathrm{T}}$.

During the formation processes of practical multi-agent systems, the leaders, located in some key positions (such as the centers or boundaries), grasp the global formation information that will be transmitted to its followers immediately. Accordingly, the followers can be in contact with their neighbors instantly and exchange their state information locally. As time goes on, all the followers can finally converge to the convex hull of the leaders′ positions, then the expected formation can be achieved. From this kind of formation idea[12, 19−21], one gives the following formation definition.

Definition 1. A formation $(\boldsymbol{F}, W)$ of N agents consists of a time-dependent column vector $\boldsymbol{F} = [f_1, f_2, \cdots, f_{\ell}]^{\mathrm{T}} \in \mathbf{R}^{\ell}$ $(\ell \le N)$ with $f_i \in \mathbf{R}$ (which can be considered as the whole formation shape function), representing the global formation information, and a time-independent nonnegative matrix $W = [W_{\ell+1}^{\mathrm{T}}, \cdots, W_N^{\mathrm{T}}]^{\mathrm{T}} \in \mathbf{R}^{(N-\ell)\ell}$, representing the local formation information, with the property that the entry sum of matrix $\boldsymbol{W_i} = (W_i^1, \cdots, W_i^{\ell})^{\mathrm{T}} \in \mathbf{R}^{\ell}$ $(i \in N^f)$ is 1 and the ith entry of W_i is zero. If there exists an $\mathbf{R}^n$-valued function $f_c(t)$ (which can be considered as the whole tracking object function) such that $x_i \to f_i + f_c$ for $i \in N^{\ell}$ and $x_i \to \boldsymbol{W_i}\boldsymbol{x}^{\ell}$ for $i \in N^f$ as $t \to \infty$, then we say that system (1) solves the formation problem. Especially, the problem is classified as a time-invariant formation (TIF) problem, a time-varying formation (TVF) problem and a time-varying formation for trajectory tracking (TVFT) problem, respectively, if both the formation shape and the tracking object are constants ($\dot{f}_i = 0$ with $\dot{f}_c = 0$), the formation shape is time-varying while the tracking object is a constant ($\dot{f}_i \neq 0$ with $\dot{f}_c = 0$), and both the formation shape and tracking object are time-varying ($\dot{f}_i \neq 0$ with $\dot{f}_c \neq 0$), respectively.

Vector F defines the basic frame of the formation that will be formed by the leaders and the non-negative matrix W specifies the local-state restrictions faced by the followers in relation to their leaders. W therefore determines the space distribution of the followers which, as a result of W's unit row entry sum property, must be a convex region. The column vector $\boldsymbol{F_c}(t) = [f_c(t), f_c(t), \cdots, f_c(t)]^{\mathrm{T}} \in \mathbf{R}^{\ell}$, on the other hand, determines the state of the formation as a whole and could therefore be dependent on both the initial states of the motions and on the driving force of any external inputs that are used to guide the group of agents into their prescribed trajectories.

Let $\mathcal{G} = (V, \varepsilon, A)$ be a weighted digraph of order N with nodes $V = \{1, 2, \cdots, N\}$, arcs $\varepsilon \subseteq V \times V$ and a weighted adjacency $A = (a_{ij}) \in \mathbf{R}^{N\times N}$, where $a_{ii} = 0$ and $a_{ij} \ge 0$ for all $i \neq j$. In the present context, $a_{ij} > 0$ if and only if there is an edge from vertex j to vertex i and a diagonal matrix $D = \mathrm{diag}\{d_1, d_2, \cdots, d_N\} \in \mathbf{R}^{N\times N}$ with $d_i = \sum_{j\in N_i} a_{ij}$ for $i = 1, 2, \cdots, N$ is called a degree matrix of $\mathcal{G}$, where $N_i = \{j \in V : (i, j) \in \varepsilon\}$ is the set of all the neighbors of node i. Matrix $L = D - A \in \mathbf{R}^{N\times N}$ is called the Laplacian matrix of the weighted digraph $\mathcal{G}$ and a directed tree is a digraph with N nodes and $N-1$ edges such that there is a directed path from the root vertex to every other vertex. A spanning tree of a digraph, on the other hand, is a subgraph that is a directed tree with the same vertex set[28].

Since state x_i of agent i may be unobservable and one usually observes a nonlinear function $h(x_i)$ of state x_i, the following nonlinear function class is introduced.

Definition 2[30]**.** A nonlinear continuous function $h : \mathbf{R} \to \mathbf{R}$ is said to belong to the acceptable nonlinear coupling function class, denoted by $NCF(\Delta, \alpha, \beta)$, if there exist three nonnegative scalars Δ, α and β, such that $h(x) - \Delta x$ satisfies

$$\alpha\,|x-y| \le |h(x) - h(y) - \Delta(x-y)| \le \beta\,|x-y|$$

for all $x, y \in \mathbf{R}$.

Intuitively, the oscillatory amplitudes of the (nonlinear) functions $h \in NCF(\Delta, \alpha, \beta)$ are being restricted by the linear functions Δx. Clearly, class $NCF(\Delta, \alpha, \beta)$ contains all the continuously differentiable functions $h(\cdot) : \mathbf{R}^n \to \mathbf{R}^n$ such that $|h'(x) - \Delta| \in [\alpha, \beta]$. There are lots of these nonlinear functions, take $h(x) = x + 0.3\sin x$ as an example, it is easy to verify that $h(x) \in NCF(\Delta, 0.2, 0.8)$ by taking $\Delta = 0.5$.

2 A stable theory of stochastic delay differential equations

In this section, we state a result on the stability of the n-dimensional stochastic differential delay equation[31]

$$\mathrm{d}\boldsymbol{x}(t) = f(t, \boldsymbol{x}(t), \boldsymbol{x}(t-\tau))\mathrm{d}t + g(t, \boldsymbol{x}(t), \boldsymbol{x}(t-\tau))\mathrm{d}\boldsymbol{w}(t) \tag{2}$$

at $t \ge 0$ with initial data $x(t) = \xi(t)$ for $-\tau \le t \le 0$. Here, $f : \mathbf{R}_+ \times \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^n$ and $g : \mathbf{R}_+ \times \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^{n\times m}$ are locally Lipschitz continuous functions that satisfy the linear growing conditions and $\boldsymbol{w}(t) = (w_1(t), \cdots, w_m(t))^{\mathrm{T}}$

is the m-dimensional Brownian motion that is defined in a complete probability space $(\Omega, \mathscr{F}, P)$ with natural filtration $\{\mathscr{F}_t\}_{t\geq 0}$ and time delay $\tau > 0$. If $\xi(s) \in L^2_{\mathscr{F}_0}([-\tau, 0]; \mathbf{R}^n)$ and the family of $\mathbf{R}^n$-valued stochastic processes such that $\xi(s)$ is $\mathscr{F}_0$-measurable at every second with $\int_{-\tau}^{0} E|\xi(s)|^2 \mathrm{d}s < \infty$ for $-\tau \leq s < 0$, then equation (2) has a unique solution $x(t,\xi)$ which is trivial (i.e. $x(t,0) \equiv 0$) for the particular case when $f(t,0,0) = 0$ and $g(t,0,0) = 0$.

Definition 3[31]**.** The trivial solution of (2) is said to be exponentially stable in mean square (respectively, almost surely exponentially stable) if

$$\lim_{t\to\infty} \sup \frac{1}{t} \ln(\mathrm{E}|x(t,\xi)|^2) < 0$$

(respectively

$$\lim_{t\to\infty} \sup \frac{1}{t} \ln|x(t,\xi)| < 0 \qquad \text{a.s.})$$

for all initial data $\xi \in L^2_{\mathscr{F}_0}([-\tau, 0]; \mathbf{R}^n)$.

Theorem 1. Assume there exists a constant $\gamma > 0$ such that

$$2\boldsymbol{x}^{\mathrm{T}} f(t,\boldsymbol{x},\boldsymbol{x}) \leq -\gamma|\boldsymbol{x}|^2 \text{ for all } (t,\boldsymbol{x}) \in \mathbf{R}_+ \times \mathbf{R}^n$$

and nonnegative constants $\theta_1, \theta_2, \theta_3$ and α_1, α_2 such that

$$\begin{cases} |f(t,\boldsymbol{x},\boldsymbol{x}) - f(t,\boldsymbol{x},\boldsymbol{y})| \leq \theta_1|\boldsymbol{x}-\boldsymbol{y}| \\ |f(t,\boldsymbol{x},\boldsymbol{y})| \leq \theta_2|\boldsymbol{x}| + \theta_3|\boldsymbol{y}| \\ \mathrm{tr}\,[g^{\mathrm{T}}(t,\boldsymbol{x},\boldsymbol{y})g(t,\boldsymbol{x},\boldsymbol{y})] \leq \alpha_1|\boldsymbol{x}|^2 + \alpha_2|\boldsymbol{y}|^2 \end{cases}$$

for all $(t,\boldsymbol{x},\boldsymbol{y}) \in \mathbf{R}_+ \times \mathbf{R}^n \times \mathbf{R}^n$. If the communication delay $\tau < \tau^* = \frac{\sqrt{\theta_1^2(\alpha_1+\alpha_2)^2 + (\theta_2^2+\theta_3^2)(\gamma-\alpha_1-\alpha_2)^2} - \theta_1(\alpha_1+\alpha_2)}{4\theta_1(\theta_2^2+\theta_3^2)}$, then the trivial solution of (2) is both exponentially stable in mean square and almost surely exponentially stable.

Proof. Let $\boldsymbol{x}(t) = \boldsymbol{x}(t,\xi)$ be the unique solution of (2) for the initial value ξ. By the conditions of $\tau < \tau^*$ in this theorem, one yields

$$\gamma > (\alpha_1+\alpha_2) + 2\theta_1\sqrt{2\tau\left(2\tau\left[\theta_2^2+\theta_3^2\right] + \alpha_1+\alpha_2\right)}$$

Thus one can take

$$\delta = \sqrt{2\tau(2\tau(\theta_2^2+\theta_3^2) + \alpha_1+\alpha_2)} \tag{3}$$

such that

$$\begin{aligned}\gamma = {} & (\alpha_1+\varepsilon) + \alpha_2\mathrm{e}^{\varepsilon\tau} + \theta_1\delta + \tfrac{\theta_1}{\delta}[2\tau(2\tau(\theta_2^2+\alpha_1)\mathrm{e}^{\varepsilon\tau} + \\ & 2\tau(2\tau(\theta_3^2+\alpha_2)\mathrm{e}^{2\varepsilon\tau}]\end{aligned} \tag{4}$$

for some chosen $\varepsilon > 0$. Then we define the Lyapunov function

$$V(t) = \mathrm{e}^{\varepsilon t}\boldsymbol{x}(t)^{\mathrm{T}}\boldsymbol{x}(t)$$

and use Ito′s formula to obtain

$$\begin{aligned}\mathrm{d}V(t) = {} & V_t\mathrm{d}t + V_x\mathrm{d}x + (1/2)\mathrm{tr}(g^{\mathrm{T}}V_{xx}g)\mathrm{d}t = \\ & [\varepsilon\, \mathrm{e}^{\varepsilon t}\boldsymbol{x}^{\mathrm{T}}(t)\boldsymbol{x}(t) + 2\mathrm{e}^{\varepsilon t}\boldsymbol{x}^{\mathrm{T}}(t)f(t,\boldsymbol{x}(t),x(t-\tau)) + \\ & \mathrm{e}^{\varepsilon t}\mathrm{tr}(g^{\mathrm{T}}(t,\boldsymbol{x}(t),\boldsymbol{x}(t-\tau))g(t,\boldsymbol{x}(t),\boldsymbol{x}(t-\tau))]\mathrm{d}t + \\ & 2\mathrm{e}^{\varepsilon t}\boldsymbol{x}^{\mathrm{T}}(t)g(t,\boldsymbol{x}(t),\boldsymbol{x}(t-\tau))\mathrm{d}\boldsymbol{w}\end{aligned} \tag{5}$$

so that we have

$$\begin{aligned}\mathrm{E}\int_0^t \mathrm{d}(V(s)) \leq {} & (-\gamma+\theta_1\delta+\alpha_1+\varepsilon)\int_0^t \mathrm{e}^{\varepsilon s}\mathrm{E}|\boldsymbol{x}(s)|^2\mathrm{d}s + \\ & \alpha_2\int_0^t \mathrm{e}^{\varepsilon s}\mathrm{E}|\boldsymbol{x}(s-\tau)|^2\mathrm{d}s + \\ & 2\mathrm{E}\int_0^t \mathrm{e}^{\varepsilon s}\boldsymbol{x}(s)^{\mathrm{T}}g\mathrm{d}\boldsymbol{w} + \\ & \tfrac{\theta_1}{\delta}\int_0^t \mathrm{e}^{\varepsilon s}\mathrm{E}|\boldsymbol{x}(s)-\boldsymbol{x}(s-\tau)|^2\mathrm{d}s\end{aligned} \tag{6}$$

because of the assumption

$$\begin{aligned}& 2\boldsymbol{x}(t)^{\mathrm{T}}f(t,\boldsymbol{x}(t),\boldsymbol{x}(t-\tau)) = 2\boldsymbol{x}(t)^{\mathrm{T}}f(t,\boldsymbol{x}(t),\boldsymbol{x}(t)) + \\ & \quad 2\boldsymbol{x}(t)^{\mathrm{T}}(f(t,\boldsymbol{x}(t),\boldsymbol{x}(t-\tau)) - f(t,\boldsymbol{x}(t),\boldsymbol{x}(t))) - \leq \\ & \quad \gamma|\boldsymbol{x}(t)|^2 + \theta_1(\delta|\boldsymbol{x}(t)|^2 + \tfrac{1}{\delta}|\boldsymbol{x}(t)-\boldsymbol{x}(t-\tau)|^2)\end{aligned}$$

where $\mathrm{tr}(g^{\mathrm{T}}g) \leq (\alpha_1|\boldsymbol{x}(t)|^2 + \alpha_2|\boldsymbol{x}(t-\tau)|^2)$. It now follows (from the conditions of this theorem) that $\boldsymbol{x}(t)$ is integrable in mean square and $g(t,\boldsymbol{x}(t),\boldsymbol{x}(t-\tau))$ satisfies the linear growing conditions and so $g(t,\boldsymbol{x}(t),\boldsymbol{x}(t-\tau))$ is also integrable in mean square. Hence, by Ito′s isometry, we have $\mathrm{E}\int_0^t \mathrm{e}^{\varepsilon s}\boldsymbol{x}(s)^{\mathrm{T}}g\mathrm{d}\boldsymbol{w} = 0$ and deduce that

$$\begin{aligned}\mathrm{E}(V(t)) \leq {} & \mathrm{E}(V(0)) + \alpha_2\int_0^t \mathrm{e}^{\varepsilon s}\mathrm{E}|\boldsymbol{x}(s-\tau)|^2\mathrm{d}s + \\ & (-\gamma+\theta_1\delta+\alpha_1+\varepsilon)\int_0^t \mathrm{e}^{\varepsilon s}\mathrm{E}|\boldsymbol{x}(s)|^2\mathrm{d}s + \\ & \tfrac{\theta_1}{\delta}\int_0^t \mathrm{e}^{\varepsilon s}\mathrm{E}|\boldsymbol{x}(t)-\boldsymbol{x}(s-\tau)|^2\mathrm{d}s\end{aligned} \tag{7}$$

for all $t \geq 0$. In particular, for $t \geq \tau$, we have

$$\begin{aligned}& \int_0^t \mathrm{e}^{\varepsilon s}\mathrm{E}|\boldsymbol{x}(s-\tau)|^2\mathrm{d}s = \\ & \quad \int_0^\tau \mathrm{e}^{\varepsilon s}\mathrm{E}|\boldsymbol{x}(s-\tau)|^2\mathrm{d}s + \int_\tau^t \mathrm{e}^{\varepsilon s}\mathrm{E}|\boldsymbol{x}(s-\tau)|^2\mathrm{d}s \leq \\ & \quad \mathrm{e}^{\varepsilon\tau}\int_{-\tau}^0 \mathrm{E}|\xi(s)|^2\mathrm{d}s + \mathrm{e}^{\varepsilon\tau}\int_0^{t-\tau}\mathrm{E}|\boldsymbol{x}(s)|^2\mathrm{d}s \leq \\ & \quad c_1\mathrm{e}^{\varepsilon\tau} + \mathrm{e}^{\varepsilon\tau}\int_0^t \mathrm{E}|\boldsymbol{x}(s)|^2\mathrm{d}s\end{aligned} \tag{8}$$

for some $c_1 = \int_{-\tau}^0 \mathrm{E}|\xi(s)|^2\mathrm{d}s$ and so

$$\mathrm{E}\left|\int_{s-\tau}^s g\mathrm{d}\boldsymbol{w}(r)\right|^2 \leq \int_{s-\tau}^s \mathrm{E}[\mathrm{tr}(g^{\mathrm{T}}g)]\mathrm{d}r$$

for all $s \geq \tau$ (because $g(t,\boldsymbol{x}(t),\boldsymbol{x}(t-\tau))$ is integrable in mean square and, again, because of Ito′s Isometry). Hölder′s inequality and the model assumptions now imply that

$$\begin{aligned}& \mathrm{E}|\boldsymbol{x}(s)-\boldsymbol{x}(s-\tau)|^2 = \mathrm{E}\left|\int_{s-\tau}^s \mathrm{d}\boldsymbol{x}(r)\right|^2 \leq \\ & \quad 2\mathrm{E}\left|\int_{s-\tau}^s f(r,\boldsymbol{x}(r),\boldsymbol{x}(r-\tau))\mathrm{d}r\right|^2 + \\ & \quad 2\mathrm{E}\left|\int_{s-\tau}^s g\mathrm{d}\boldsymbol{w}(r)\right|^2 \leq \\ & \quad 2\tau\mathrm{E}\int_{s-\tau}^s |f(r,\boldsymbol{x}(r),\boldsymbol{x}(r-\tau))|^2\mathrm{d}r + \\ & \quad 2\int_{s-\tau}^s \mathrm{E}\left[\mathrm{tr}\left(g^{\mathrm{T}}g\right)\right]\mathrm{d}r \leq \\ & \quad 2\tau\mathrm{E}\left(\int_{s-\tau}^s (\theta_2|\boldsymbol{x}(r)| + \theta_3|\boldsymbol{x}(r-\tau)|)^2\mathrm{d}r\right) + \\ & \quad 2\int_{s-\tau}^s \left(\alpha_1\mathrm{E}|\boldsymbol{x}(r)|^2 + \alpha_2\mathrm{E}|\boldsymbol{x}(r-\tau)|^2\right)\mathrm{d}r = \\ & \quad 2\left(2\tau\theta_2^2+\alpha_1\right)\int_{s-\tau}^s \mathrm{E}|\boldsymbol{x}(r)|^2\mathrm{d}r + \\ & \quad 2\left(2\tau\theta_3^2+\alpha_2\right)\int_{s-\tau}^s \mathrm{E}|\boldsymbol{x}(r-\tau)|^2\mathrm{d}r\end{aligned} \tag{9}$$

and that, in particular,

$$\begin{aligned}& \int_0^t \mathrm{e}^{\varepsilon s}\mathrm{E}|\boldsymbol{x}(s)-\boldsymbol{x}(s-\tau)|^2\mathrm{d}s \leq \\ & \quad c_2 + 2\left(2\tau\theta_2^2+\alpha_1\right)\int_\tau^t \mathrm{e}^{\varepsilon s}\int_{s-\tau}^s \mathrm{E}|\boldsymbol{x}(r)|^2\mathrm{d}r\mathrm{d}s + \\ & \quad 2\left(2\tau\theta_3^2+\alpha_2\right)\int_\tau^t \mathrm{e}^{\varepsilon s}\int_{s-\tau}^s \mathrm{E}|\boldsymbol{x}(r-\tau)|^2\mathrm{d}r\mathrm{d}s\end{aligned} \tag{10}$$

for all $t \geq \tau$ for some $c_2 = \int_0^\tau e^{\varepsilon s} E\left|\boldsymbol{x}(s) - \boldsymbol{x}(s-\tau)\right|^2 ds$. Since this last inequality can be written as

$$\begin{aligned}&\int_0^t e^{\varepsilon s} E\left|\boldsymbol{x}(s) - \boldsymbol{x}(s-\tau)\right|^2 ds \leq \\ &\quad c_2 + 2(2\tau\theta_2^2 + \alpha_1)\tau e^{\varepsilon\tau} \int_0^t e^{\varepsilon s} E\left|\boldsymbol{x}(s)\right|^2 ds + \\ &\quad 2(2\tau\theta_3^2 + \alpha_2)\tau e^{2\varepsilon\tau}\left(c_1 + \int_0^t e^{\varepsilon s} E\left|\boldsymbol{x}(s)\right|^2 ds\right)\end{aligned} \tag{11}$$

by using the facts that

$$\begin{aligned}&\int_\tau^t e^{\varepsilon s}\int_{s-\tau}^s E\left|\boldsymbol{x}(r)\right|^2 dr ds = \\ &\quad \int_0^t E\left|\boldsymbol{x}(r)\right|^2 (\int_{r\vee\tau}^{(r+\tau)\wedge t} e^{\varepsilon s} ds) dr \leq \\ &\quad \tau e^{\varepsilon\tau}\int_0^t e^{\varepsilon s} E\left|\boldsymbol{x}(s)\right|^2 ds\end{aligned}$$

and

$$\begin{aligned}&\int_\tau^t e^{\varepsilon s}\int_{s-\tau}^s E\left|\boldsymbol{x}(r-\tau)\right|^2 dr ds = \\ &\quad \int_0^t E\left|\boldsymbol{x}(r-\tau)\right|^2 \left(\int_{r\vee\tau}^{(r+\tau)\wedge t} e^{\varepsilon s} ds\right) dr \leq \\ &\quad \tau e^{\varepsilon\tau}\int_0^t e^{\varepsilon s} E\left|\boldsymbol{x}(s-\tau)\right|^2 ds \leq \\ &\quad c_1\tau d^{2\varepsilon\tau} + \tau e^{2\varepsilon\tau}\int_0^t e^{\varepsilon s} E\left|\boldsymbol{x}(s)\right|^2 ds\end{aligned}$$

substitution of (8) and (11) into (7) gives

$$\begin{aligned}&E[V(t)] \leq E[V(0)] + c_1\alpha_2 e^{\varepsilon\tau} + \\ &\quad \theta_1(1/\delta)\left[c_2 + 2c_1\left(2\tau\theta_3^2 + \alpha_2\right)\tau e^{2\varepsilon\tau}\right] + \\ &\quad \left(\int_0^t e^{\varepsilon s} E\left|\boldsymbol{x}(s)\right|^2 ds\right)\{(-\gamma + \alpha_1 + \varepsilon + \theta_1\delta) + \\ &\quad \alpha_2 e^{\varepsilon\tau} + \theta_1(1/\delta)[2\tau(2\tau\theta_2^2 + \alpha_1)e^{\varepsilon\tau} + \\ &\quad 2\tau(2\tau\theta_3^2 + \alpha_2)e^{2\varepsilon\tau}]\}\end{aligned}$$

which can be written, by using (4), as

$$\begin{aligned}E[V(t)] \leq\ & E[V(0)] + \frac{\theta_1}{\delta}[c_2 + 2c_1\left(2\tau\theta_3^2 + \alpha_2\right)\tau e^{2\varepsilon\tau}] + \\ & c_1\alpha_2 e^{\varepsilon\tau} := c_3\end{aligned}$$

The definition $V(t) = e^{\varepsilon t}\boldsymbol{x}(t)^{\rm T}(t) = e^{\varepsilon t}\left|\boldsymbol{x}(t)\right|^2$ now shows that

$$E\left|\boldsymbol{x}(t)\right|^2 \leq c_3 e^{-\varepsilon t} \tag{12}$$

for all $t \geq \tau$ and hence that

$$\lim_{t\to\infty}\sup\frac{1}{t}\ln\left(E\left|\boldsymbol{x}(t)\right|^2\right) \leq -\varepsilon \tag{13}$$

and the trivial solution of (2) is thus exponentially stable in mean square.

In the same way, the trivial solution of (2) can also be shown to be almost surely exponentially stable. In fact,

$$\begin{aligned}&E\left(\sup_{k\tau\leq t\leq(k+1)\tau}\left|\boldsymbol{x}(t)\right|^2\right) \leq \\ &\quad 3E\left|\boldsymbol{x}(k\tau)\right|^2 + 3E\left|\int_{k\tau}^{(k+1)\tau} f\left(t, \boldsymbol{x}(t), \boldsymbol{x}(t-\tau)\right) dt\right|^2 + \\ &\quad 3E\left|\int_{k\tau}^{(k+1)\tau} g\left(t, \boldsymbol{x}(t), \boldsymbol{x}(t-\tau)\right) d\boldsymbol{w}(t)\right|^2 \leq \\ &\quad 3E\left|\boldsymbol{x}(k\tau)\right|^2 + 3\int_{k\tau}^{(k+1)\tau}\left[\left(2\tau\theta_2^2 + \alpha_1\right) E\left|\boldsymbol{x}(t)\right|^2 + \right. \\ &\quad \left.\left(2\tau\theta_3^2 + \alpha_2\right) E\left|\boldsymbol{x}(t-\tau)\right|^2\right] dt \leq \\ &\quad 3c_3\left[e^{-\varepsilon k\tau} + \left(2\tau\theta_2^2 + \alpha_1\right)\int_{k\tau}^{(k+1)\tau} e^{-\varepsilon t} dt + \right. \\ &\quad \left.\left(2\tau\theta_3^2 + \alpha_2\right)\int_{k\tau}^{(k+1)\tau} e^{-\varepsilon(t-\tau)} dt\right] \leq c_4 e^{-\varepsilon k\tau}\end{aligned}$$

for each $k = 2, 3, \cdots$ and for some

$$\begin{aligned}c_4 = 3c_3\Bigg[&1 + \frac{\left(2\tau\theta_2^2 + \alpha_1\right)\left(1 - e^{-\varepsilon\tau}\right)}{\varepsilon} + \\ &\frac{\left(2\tau\theta_3^2 + \alpha_2\right)\left(e^{\varepsilon\tau} - 1\right)}{\varepsilon}\Bigg]\end{aligned}$$

Let $\eta \in (0, \varepsilon)$ be arbitrary; Chebyshev′s inequality gives

$$\begin{aligned}&P\left(\sup_{k\tau\leq t\leq(k+1)\tau}\left|\boldsymbol{x}(t)\right| \geq e^{(\eta-\varepsilon)k\tau/2}\right) \leq \\ &\quad e^{(\varepsilon-\eta)k\tau} E\left(\sup_{k\tau\leq t\leq(k+1)\tau}\left|\boldsymbol{x}(t)\right|^2\right) := b_k \leq c_4 e^{-\eta k\tau}\end{aligned}$$

from which it follows that $\sum b_k < \infty$. The Borel-Cantelli lemma then guarantees that

$$\sup_{k\tau\leq t\leq(k+1)\tau}\left|\boldsymbol{x}(t)\right| \leq c_4 e^{-(\varepsilon-\eta)k\tau/2}$$

holds for all but finitely many k and hence that

$$\frac{1}{t}\ln\left|\boldsymbol{x}(t)\right| \leq -\frac{\varepsilon - \eta}{2}, \quad \text{a.s.}$$

for $k\tau \leq t \leq (k+1)\tau$ when k is large enough and so

$$\lim_{t\to\infty}\sup\frac{1}{t}\ln\left|x(t)\right| \leq -\frac{\varepsilon}{2}, \quad \text{a.s.}$$

in the limit as $\eta \to 0$. The trivial solution of (2) is thus almost surely exponentially stable. □

3 Formation control protocols

In this section, we present the precise control protocols for three particular formations: time-invariant formations (Corollary 1), time-varying formations (Corollary 2) and time-varying formations for trajectory tracking (Theorem 3). In order to simplify this problem, in the following, one will firstly give the results of time-varying consensus, under which the three formations can be subsequently derived.

3.1 Time-varying consensus

The simplest formation is the time-varying consensus problem in which we assume that $f_0 = f_1 = f_2 = \cdots = f_\ell$ with $f_0 = a_0$ for some formation velocity a_0 and $\ell = N$. A multi-agent system is said to have attained consensus in mean square and consensus almost surely if $\lim_{t\to\infty}\sup\frac{1}{t}\ln E\left|x_i - f_0\right|^2 < 0$ and $\lim_{t\to\infty}\sup\frac{1}{t}\ln\left|x_i - f_0\right| < 0$ respectively for all $i = 1, 2, \cdots, N$ and every agent is able to attain velocity a_0 because every agent is a leader in this case. Any discrepancies between the specified formation information and the agents′ actual states, however, are only detectable to the leaders of the leaders who are also responsible for pinning the other leaders to attain the expected formation. More precisely, the information available to the ith agent with respect to its neighbors is

$$y_{ij} = a_{ij}\left[h\left(x_j(t-\tau)\right) - h\left(x_i(t-\tau)\right)\right]$$

($j \in N_i$) for some time delay $\tau > 0$, connection weight a_{ij} between agents i and j and continuous function $h: \mathbf{R} \to \mathbf{R}$

is strictly increasing. Without loss of generality, we assume $h(0) = 0$. The state deviations available to the ith agent is

$$z_i = b_i[h(x_i(t)) - h(f_0(t))]$$

for some $b_i \geq 0$ ($b_i > 0$ if and only if agent i is the leader of the leaders′ set).

Note that y_{ij} describes the information states deviations between agent i and its neighbor $j \in N_i$ estimated by agent i at time t, and τ is the delays generated during the estimation process. However, for agent i with $b_i > 0$ (the leader of the leaders′ set), there is no delay in z_i since the control input is available directly to such a leader.

Let the noise perturbation intensity be

$$\phi_{ij} = \sigma_{ij}\left[h\left(x_j(t-\tau)\right) - h\left(x_i(t-\tau)\right)\right]$$

and $\psi_i = \rho_i[h(x_i(t)) - h(f_0(t))]$ for some constants σ_{ij} such that $\sigma_{ij} > 0$ for $j \in N_i$, $\sigma_{ij} = 0$ for $j \notin N_i$ ($j \neq i$) and $\sigma_{ii} = \sum_{j \in N_i} \sigma_{ij}$. $\rho_i \geq 0$ for some $\rho_i > 0$ when $b_i > 0$. Then the noise-perturbed time-varying consensus protocol can be designed as

$$u_i = a_0 + \sum_{j \in N_i} y_{ij} - z_i + \left(\sum_{j \in N_i} \phi_{ij} - \psi_i\right)\boldsymbol{w} \tag{14}$$

for some one-dimensional Brownian motion $w(t)$ with (one-dimensional) white noise $\dot{w}$, as defined in Section 2.

Here, as a given control input, it is assumed that the formation velocity a_0 is available to all the agents located in the leaders′ set (which actually contains a very little number of the whole formation agents). Nevertheless, it is not available to any follower agent, which can be seen in three formation protocols (see (18)).

This protocol is superior to many of the currently existing ones because it uses a nonlinear control strategy that inevitably models real-world formation control problems better than the linear ones[11, 15]. Actually, in some cases, state x_i of the ith agent may be unobservable due to the signal conversion and the transmission of measurement data through physical devices. Our proposed protocol, furthermore, accounts also for the unavoidable noise effects that arise from the transmission of information between the neighbors[25−27]. Finally, since most of the existing results on delayed systems usually propose the formation tracking consensus conditions which are time-delay independent, which may be more conservative than those of time-delay dependent consensus conditions. Hence, this paper aims to deduce the time-delay dependent consensus conditions for the multi-agent systems formation under noise disturbance and give the corresponding upper bound of the communication delay, which is more challengeable and meaningful to open out the intrinsic relationships between the asymmetric communication delays and the consensus ability of multi-agent systems.

We write system (1), under the control protocol (14), in the form of an Ito stochastic delay differential equation

$$\mathrm{d}x_i = \left[a_0 + \sum_{j \in N_i} y_{ij} - z_i\right]\mathrm{d}t + \left[\sum_{j \in N_i} \phi_{ij} - \psi_i\right]\mathrm{d}\boldsymbol{w} \tag{15}$$

and we have the following result.

Theorem 2. If $h \in NCF(\Delta, \alpha, \beta)$, $\tau < \tau^* = \frac{\sqrt{\theta_1^2(\alpha_1+\alpha_2)^2 + \left(\theta_2^2+\theta_3^2\right)(\gamma-\alpha_1-\alpha_2)^2} - \theta_1(\alpha_1+\alpha_2)}{4\theta_1\left(\theta_2^2+\theta_3^2\right)}$, where α_1, α_2, θ_1, θ_2, θ_3 and γ are some positive constants related to some system parameters, and digraph $\mathcal{G}$ contains a spanning directed tree with the root node located in leaders′ group, then all the agents of system (15) will attain consensus almost surely.

Proof. Taking $e_i = x_i - f_0$, we have the error system

$$\begin{aligned}\mathrm{d}e_i = \{ &\sum_{j \in N_i} a_{ij}\left[h\left(x_j\left(t-\tau\right)\right) - h\left(x_i\left(t-\tau\right)\right)\right] - \\ &b_i\left[h\left(x_i\left(t\right)\right) - h\left(f_0\left(t\right)\right)\right]\}\mathrm{d}t + \\ &\{\sum_{j \in N_i} \sigma_{ij}\left[h\left(x_j\left(t-\tau\right)\right) - h\left(x_i\left(t-\tau\right)\right)\right] - \\ &\rho_i\left[h\left(x_i\left(t\right)\right) - h\left(f_0\left(t\right)\right)\right]\}\mathrm{d}w\end{aligned} \tag{16}$$

which can be written as

$$\begin{aligned}&\mathrm{d}\boldsymbol{e}(t) = \{(A-D)\left[\boldsymbol{H}\left(\boldsymbol{x}(t-\tau)\right) - \boldsymbol{H}\left(f_0(t-\tau)\otimes I_N\right)\right] - \\ &B\left[\boldsymbol{H}\left(\boldsymbol{x}(t)\right) - \boldsymbol{H}\left(f_0(t)\otimes I_N\right)\right]\}\,\mathrm{d}t + \\ &\quad\{(A_\sigma - D_\sigma)\left[\boldsymbol{H}\left(\boldsymbol{x}(t-\tau)\right) - \boldsymbol{H}\left(f_0(t-\tau)\otimes I_N\right)\right] - \\ &\quad B_\sigma\left[\boldsymbol{H}\left(\boldsymbol{x}(t)\right) - \boldsymbol{H}\left(f_0(t)\otimes I_N\right)\right]\}\,\mathrm{d}\boldsymbol{w}\end{aligned}$$

where $A = (a_{ij})_{N\times N}$, $D = \mathrm{diag}\{d_1, d_2, \cdots, d_N\}$, $B = \mathrm{diag}\{b_1, b_2, \cdots, b_N\}$, $\boldsymbol{e} = [e_1, \cdots, e_N]^{\mathrm{T}}$, $\boldsymbol{x} = [x_1, \cdots, x_N]^{\mathrm{T}}$, $A_\sigma = (\sigma_{ij})_{N\times N}$, $D_\sigma = \mathrm{diag}\{\sigma_{11}, \cdots, \sigma_{NN}\}$, $B_\sigma = \mathrm{diag}\{\rho_{11}, \cdots, \rho_{NN}\}$, $\boldsymbol{H}(e) = [h(e_1), \cdots, h(e_N)]^{\mathrm{T}}$.

Obviously, according to the definitions of function h and $\boldsymbol{w}(t)$, it is easy to verify that the stochastic differential delay equation (16) is well defined and satisfies the conditions proposed for equation (2). Thus the Ito formula can be used hereinafter.

Let $L = D - A$ and $L_\sigma = D_\sigma - A_\sigma$. Then L is the Laplacian matrix associated with digraph $\mathcal{G}$ and L_σ has the same properties as L. We now prove that $L + B$ is positive stable. Since $\mathcal{G}$ contains a directed spanning tree with the root node located in leaders′ group (assume $r = 1$), L can be written in the Frobenius normal form[32] as

$$L = \begin{bmatrix} L_{11} & 0 \\ L_{21} & L_{22} \end{bmatrix}$$

for some one-dimensional zero matrix L_{11} that corresponds to the root vertex $r = 1$ and a nonsingular matrix L_{22} whose eigenvalues have positive real parts (by Lemma 2 in [17]). $B = \mathrm{diag}\{B_1, B_2\}$ is then a block diagonal matrix with $B_1 = b_1 > 0$ and $B_2 = \mathrm{diag}\{b_2, \cdots, b_N\}$ and so $L + B$ is positive stable because $b_1 > 0$ and $L_{11} + B_1$ is positive stable. Let $(L+B)^s = [(L+B) + (L+B)^{\mathrm{T}}]/2$ and $(L+B)^a = [(L+B) - (L+B)^{\mathrm{T}}]/2$. It is easy to see that $(L+B)^s$ is symmetric and positive definite and that $(L+B)^a$ is anti-symmetric. It thus follows (from matrix analysis[19]) that $[(L+B)^a]^{\mathrm{T}}[(L+B)^a]$ is symmetric with nonnegative eigenvalues and so

$$\begin{aligned}&2\boldsymbol{e}^{\mathrm{T}}(t)\left(A-D-B\right)\left[\boldsymbol{H}\left(\boldsymbol{x}(t)\right) - \boldsymbol{H}\left(f_0(t)\otimes I_N\right)\right] = \\ &\quad -2\Delta\boldsymbol{e}^{\mathrm{T}}(t)\left(L+B\right)^s\boldsymbol{e}(t) - 2\boldsymbol{e}^{\mathrm{T}}(t)\left(L+B\right)^s \times \\ &\quad \left[\boldsymbol{H}\left(\boldsymbol{x}(t)\right) - \boldsymbol{H}\left(f_0(t)\otimes I_N\right) - \Delta\boldsymbol{e}(t)\right] - \\ &\quad 2\boldsymbol{e}^{\mathrm{T}}(t)\left(L+B\right)^a\left[\boldsymbol{H}\left(\boldsymbol{x}(t)\right) - \boldsymbol{H}\left(f_0(t)\otimes I_N\right)\right] \leq \\ &\quad -2(\Delta+\alpha)\lambda_{\min}\left(\left(L+B\right)^s\right)\left|\boldsymbol{e}(t)\right|^2 + \\ &\quad \left(\left(1/\mu\right)\lambda_{\max}\left(-[(L+B)^a]^2\right) + \mu(\Delta+\beta)^2\right)\left|\boldsymbol{e}(t)\right|^2 \leq \\ &\quad -\gamma\left|\boldsymbol{e}(t)\right|^2\end{aligned}$$

for some positive scalars μ and

$$\gamma = 2(\Delta+\alpha)\lambda_{\min}\left((L+B)^s\right) - \mu(\Delta+\beta)^2 - (1/\mu)\,\lambda_{\max}\left(-[(L+B)^a]^2\right) > 0$$

and all $(t,e) \in \mathbf{R}_+ \times \mathbf{R}^N$. Furthermore, we have

$$\begin{aligned}&|(A-D-B)\left[\boldsymbol{H}(\boldsymbol{x}(t)) - \boldsymbol{H}(f_0(t)\otimes I_N)\right] - \\ &\quad \{(A-D)\left[\boldsymbol{H}(\boldsymbol{x}(t-\tau)) - \boldsymbol{H}(f_0(t-\tau)\otimes I_N)\right] - \\ &\quad B\left[\boldsymbol{H}(\boldsymbol{x}(t)) - \boldsymbol{H}(f_0(t)\otimes I_N)\right]\}\big| \le \\ &\quad \|A-D\|\,|(\Delta+\beta)(\boldsymbol{x}(t)-\boldsymbol{x}(t-\tau)) + \\ &\quad (\Delta+\beta)(f_0(t-\tau)\otimes I_N - f_0(t)\otimes I_N)| \le \\ &\quad \|A-D\|\,(\Delta+\beta)\,|(\boldsymbol{x}(t)-f_0(t)\otimes I_N) - \\ &\quad (\boldsymbol{x}(t-\tau)-f_0(t-\tau)\otimes I_N)| \le \\ &\quad \|A-D\|\,(\Delta+\beta)\,|\boldsymbol{e}(t)-\boldsymbol{e}(t-\tau)|\end{aligned}$$

and

$$\begin{aligned}&|\{(A-D)\left[\boldsymbol{H}(\boldsymbol{x}(t-\tau)) - \boldsymbol{H}(f_0(t-\tau)\otimes I_N)\right] - \\ &\quad B\left[\boldsymbol{H}(\boldsymbol{x}(t)) - \boldsymbol{H}(f_0(t)\otimes I_N)\right]\}\big| \le \\ &\|B\|\,(\Delta+\beta)\,|\boldsymbol{e}(t)| + \|A-D\|\,(\Delta+\beta)\,|\boldsymbol{e}(t-\tau)|\end{aligned}$$

and also

$$\begin{aligned}&\mathrm{tr}(\{(A_\sigma-D_\sigma)\left[\boldsymbol{H}(\boldsymbol{x}(t-\tau)) - \boldsymbol{H}(f_0(t-\tau)\otimes I_N)\right] - \\ &\quad B_\sigma\left[\boldsymbol{H}(\boldsymbol{x}(t)) - \boldsymbol{H}(f_0(t)\otimes I_N)\right]\}^{\mathrm{T}} \times \\ &\quad \{(A_\sigma-D_\sigma)\left[\boldsymbol{H}(\boldsymbol{x}(t-\tau)) - \boldsymbol{H}(f_0(t-\tau)\otimes I_N)\right] - \\ &\quad B_\sigma\left[\boldsymbol{H}(\boldsymbol{x}(t)) - \boldsymbol{H}(f_0(t)\otimes I_N)\right]\} \le \\ &(\Delta+\beta)^2\left[\|B_\sigma\|\,|\boldsymbol{e}(t)| + \|A_\sigma-D_\sigma\|\,|\boldsymbol{e}(t-\tau)|\right]^2 \le \\ &2(\Delta+\beta)^2\left(\|B_\sigma\|^2\,|\boldsymbol{e}(t)|^2 + \|A_\sigma-D_\sigma\|^2\,|\boldsymbol{e}(t-\tau)|^2\right)\end{aligned}$$

It now follows from Theorem 1, by taking $\tau < \tau^*$ with $\alpha_1 = 2(\Delta+\beta)^2\|B_\sigma\|^2$, $\alpha_2 = 2(\Delta+\beta)^2\|A_\sigma - D_\sigma\|^2$, $\theta_1 = \theta_3 = (\Delta+\beta)\|A-D\|$ and $\theta_2 = (\Delta+\beta)\|B\|$, that the trivial solution of equation (16) is almost surely exponentially stable (in other words, that $\lim_{t\to\infty}\sup\frac{1}{t}\ln|x_i - f_0| < 0$ almost surely for $i = 1, 2, \cdots, N$). □

3.2 Three different formations

Since time-invariant formations (TIF) and time-varying formations (TVF) are special cases of time-varying formations for trajectory tracking (TVFT), in this subsection, we will firstly discuss the most complexed case (i.e., TVFT; see Theorem 3), and the other cases can be accordingly presented (see Corollaries 1 and 2).

At first, we consider TVFT in which the agents are expected to attain distinct limit states and the expected trajectory of the formation is determined by

$$\dot{f}_c = \varphi(t, f_c) \tag{17}$$

More precisely, our objective is to obtain $x_i \to f_i + f_c$ with $\dot{f}_i = a_i$ for $i \in N^\ell$ and $x_i \to W_i x^\ell$ for $i \in N^f$ as $t \to \infty$ almost surely in the context of the following assumptions:

1) There is a directed spanning tree rooted at r for some $1 \le r \le \ell$ in the local interaction topology of the leaders, whose dynamics are independent of those of their followers;

2) Every leader has access to the global formation information $\boldsymbol{F}$, which they will send to the followers in due course;

3) Every follower has access to the local formation information W either directly or indirectly from the leaders;

4) Some root agents of the local interaction topology are allowed to access the reference trajectory.

Technically, Assumption 3) is equivalent to the existence of a directed spanning tree with root vertices at the leaders in the (multi-agent) system′s interaction topology.

Referring to the construction idea of protocol (14), we design the formation control protocol as follows. For each $i \in N^\ell$, the global formation information with respect to its neighbors and state deviations that are available to agent i are

$$\begin{aligned}y_{ij}^\ell = \;& a_{ij}[h(x_j(t-\tau) - f_j(t-\tau)) - \\ & h(x_i(t-\tau) - f_i(t-\tau))],\ j \in N_i\end{aligned}$$

and

$$z_i^\ell = b_i^\ell[h(x_i(t) - f_i(t)) - h(f_c(t))]$$

respectively, for some $b_i^\ell \ge 0$ (with $b_i^\ell > 0$ if agent i is a leader of the leaders) and the noise disturbances are

$$\begin{aligned}\phi_{ij}^\ell = \;& \sigma_{ij}[h(x_j(t-\tau) - f_j(t-\tau)) - \\ & h(x_i(t-\tau) - f_i(t-\tau))],\ j \in N_i\end{aligned}$$

and $\psi_i^\ell = \rho_i[h(x_i(t) - f_i(t)) - h(f_c(t))]$. For each $i \in N^f$, however, the local formation information with respect to its neighbors and state deviations that are available to agent i are

$$\begin{aligned}y_{ij}^f = \;& a_{ij}[h(x_j(t-\tau) - \sum_{k\in N^\ell} W_j^k x_k(t-\tau)) - \\ & h(x_i(t-\tau) - \sum_{k\in N^\ell} W_i^k x_k(t-\tau))],\ j \in N_i\end{aligned}$$

and

$$z_i^f = b_i^f[h(x_i(t) - \sum_{k\in N^\ell} W_i^k x_k(t))]$$

respectively, for some $b_i^f \ge 0$ (with $b_i^f > 0$ if agent i is a leader of the followers) and the noise disturbances are

$$\begin{aligned}\phi_{ij}^f = \;& \sigma_{ij}[h(x_j(t-\tau) - \sum_{k\in N^\ell} W_j^k x_k(t-\tau)) - \\ & h(x_i(t-\tau) - \sum_{k\in N^\ell} W_i^k x_k(t-\tau))]\end{aligned}$$

and $\psi_i^f = \rho_i h(x_i(t) - \sum_{k\in N^\ell} W_i^k x_k(t))$. The control protocol for this case is hence

$$u_i = \begin{cases} a_i + \varphi(t, f_c) + \sum\limits_{j\in N_i} y_{ij}^\ell - z_i^\ell + (\sum\limits_{j\in N_i}\phi_{ij}^\ell - \psi_i^\ell)w, \\ \qquad i \in N^\ell \\ \sum\limits_{k\in N^l} W_i^k u_k + \sum\limits_{j\in N_i} y_{ij}^f - z_i^f + (\sum\limits_{j\in N_i}\phi_{ij}^f - \psi_i^f)w, \\ \qquad i \in N^f \end{cases} \tag{18}$$

and system (1) can be written (using control protocol (18)) in the form

$$\mathrm{d}x_i = \begin{cases} \left(a_i + \varphi(t, f_c) + \sum\limits_{j\in N_i} y_{ij}^\ell - z_i^\ell\right)\mathrm{d}t + \\ \quad \left(\sum\limits_{j\in N_i}\phi_{ij}^\ell - \psi_i^\ell\right)\mathrm{d}w,\ i \in N^\ell \\ \sum\limits_{k\in N^l} W_i^k \mathrm{d}x_k + \left(\sum\limits_{j\in N_i} y_{ij}^f - z_i^f\right)\mathrm{d}t + \\ \quad \left(\sum\limits_{j\in N_i}\phi_{ij}^f - \psi_i^f\right)\mathrm{d}w,\ i \in N^f \end{cases} \tag{19}$$

Theorem 3 (TVFT). If $h(\cdot) \in NCF(\Delta,\alpha,\beta)$, $\tau < \tau^* = \frac{\sqrt{\theta_1^2(\alpha_1+\alpha_2)^2+(\theta_2^2+\theta_3^2)(\gamma-\alpha_1-\alpha_2)^2}-\theta_1(\alpha_1+\alpha_2)}{4\theta_1(\theta_2^2+\theta_3^2)}$, where α_1, α_2, θ_1, θ_2, θ_3 and γ are some positive constants related to some system parameters, and digraph $\mathcal{G}$ contains a spanning directed tree with the root node located in leaders′ group, then system (19) solves the time-varying formation problem for trajectory tracking almost surely under assumptions 1) ~ 4).

Proof. Use the change of variables $\bar{x}_i = x_i - f_i$, $e_i = \bar{x}_i - f_c$ for $i \in N^\ell$ and $\bar{x}_i = x_i - \sum_{k\in N^\ell} W_i^k x_k$, $e_i = \bar{x}_i - 0$ for $i \in N^f$ to put system (19) into the form of equation (16). Then Theorem 2 guarantees that $e_i = x_i - f_i - f_c$ tends to zero exponentially almost surely and so $\lim_{t\to\infty}\sup\frac{1}{t}|x_i-(f_i+f_c)| < 0$ almost surely for $i \in N^\ell$ and $\lim_{t\to\infty}\sup\frac{1}{t}\left|x_i(t)-W_i x^l\right| < 0$ for $i \in N^f$. □

Corollary 1 (TIF). If both the global formation information F and the formation state F_c are time-invariant (i.e., $\dot{f}_i = 0$ and $\dot{f}_c = 0$) and the conditions of Theorem 3 hold, then system (19) with $a_i = 0$ and $\varphi(t, f_c) = 0$ solves the time-invariant formation problem almost surely under assumptions 1) ~ 3).

Corollary 2 (TVF). If the global formation information F is time-varying (i.e., $\dot{f}_i \neq 0$), the formation state F_c is time-invariant (i.e., $\dot{f}_c = 0$), and the conditions of Theorem 3 hold, then system (19) with $\varphi(t, f_c) = 0$ solves the time-varying formation problem almost surely under assumptions 1) ~ 3).

4 Numerical example

In this section, we apply the theory of Section 2 to a particular formation problem with twenty-five agents (seven leaders and eighteen followers) that are moving in a formation frame F (as shown in Fig. 1) in a 2-dimensional space with an anticipated formation that is a hexagon. Agents are assumed to be equipped with a communication system that allows them to be in contact with their neighbors instantly and exchange their state information under a certain noise disturbance.

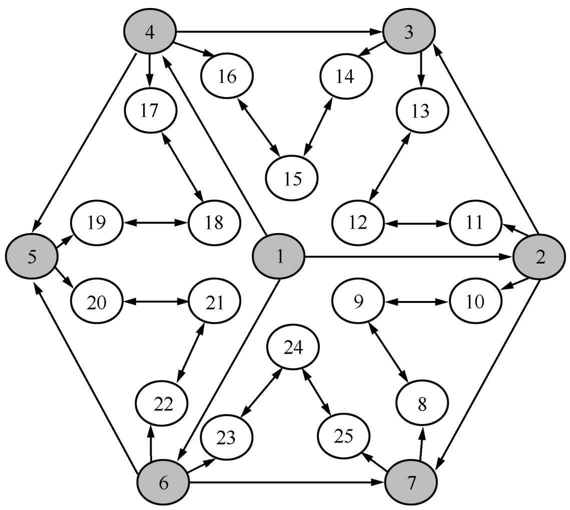

Fig. 1 The interaction topology of 25 agents

The leaders are labelled from 1 to 7 and the followers from 8 to 25 and the weighting factors are all equal (i.e. all equal to 1). Their initial positions $(x_i(0), y_i(0))^{\mathrm{T}}$ are randomly selected from the domain of $[-5,5]\times[-5,5]$. Take the system parameter $b_i = 5$ $(i = 1,2,3,4)$ and let the continuous function be $h(\boldsymbol{x}) = 2\boldsymbol{x} + \frac{0.04}{\pi}\arctan(\boldsymbol{x})$ with $\boldsymbol{x} \in \mathbf{R}^2$ so that $h(\cdot) \in NCF(1.97, 0.01, 0.05)$. The stochastic disturbance term is $g(t,\boldsymbol{x}(t),\boldsymbol{x}(t-\tau)) = \sigma f(t,\boldsymbol{x}(t),\boldsymbol{x}(t-\tau))$ with $\sigma = 0.5$.

$$\begin{cases} \boldsymbol{f}_1 = \boldsymbol{f}_2 = (0,0)^{\mathrm{T}}, \quad R_i(t) = \mathrm{sig}(\sin t)\sin t \\ \boldsymbol{f}_i = R_i(t)\left(\cos\left(\frac{(i-2)\pi}{3}\right), \sin\left(\frac{(i-2)\pi}{3}\right)\right)^{\mathrm{T}}, i = 2,\cdots,7 \\ \boldsymbol{f}_c = (2\sin(12t)+15t, 2\sin(12t)+15t)^{\mathrm{T}} \end{cases}$$

And the elements of the nonnegative matrix W are

$$\begin{cases} w_8^1 = w_8^2 = w_9^2 = w_9^7 = w_{10}^1 = w_{10}^7 = w_{11}^1 = w_{11}^3 = w_{12}^2 = \\ \quad w_{12}^3 = w_{13}^1 = w_{13}^2 = w_{14}^1 = w_{14}^4 = w_{15}^3 = w_{15}^4 = w_{16}^1 = \\ \quad w_{16}^3 = w_{17}^1 = w_{17}^5 = w_{18}^4 = w_{18}^5 = w_{19}^1 = w_{19}^4 = w_{20}^1 = \\ \quad w_{20}^6 = w_{21}^5 = w_{21}^6 = w_{22}^1 = w_{22}^5 = w_{23}^1 = w_{23}^7 = w_{24}^6 = \\ \quad w_{24}^7 = w_{25}^1 = w_{25}^6 = \frac{1}{6} \\ w_8^7 = w_9^1 = w_{10}^2 = w_{11}^2 = w_{12}^1 = w_{13}^3 = w_{14}^3 = w_{15}^1 = w_{16}^4 = \\ \quad w_{17}^4 = w_{18}^1 = w_{19}^5 = w_{20}^5 = w_{21}^1 = w_{22}^6 = w_{23}^6 = w_{24}^1 = \\ \quad w_{25}^7 = \frac{2}{3} \\ w_i^j = 0, \quad \text{in other cases} \end{cases}$$

Calculations then give $\|A-D\| = 5.5366$, $\|B\| = 5.0000$, $\lambda_{\min}((L+B)^s) = 2.4026$, $\lambda_{\max}\left(-[(L+B)^a]^2\right) = 2.5000$, $\theta_1 = \theta_3 = (\Delta+\beta)\|A-D\| = 11.1840$, $\theta_2 = (\Delta+\beta)\|B\| = 10.1000$, $\alpha_1 = 2\sigma^2\theta_2^2 = 51.005$, $\alpha_2 = 2\sigma^2\theta_3^2 = 62.5405$ and

$$\gamma = 2(\Delta+\alpha)\lambda_{\min}((L+B)^s) - \mu(\Delta+\beta)^2 - (1/\mu)\lambda_{\max}\left(-[(L+B)^a]^2\right) = 3.1267$$

with $\mu = 0.7827$. Then let the communication delay $\tau = 0.01 < \tau^* = \frac{\sqrt{\theta_1^2(\alpha_1+\alpha_2)^2+(\theta_2^2+\theta_3^2)(\gamma-\alpha_1-\alpha_2)^2}-\theta_1(\alpha_1+\alpha_2)}{4\theta_1(\theta_2^2+\theta_3^2)} = 0.0810$, thus the conditions of Theorems 3 are all satisfied.

Some simulations are then carried out for 1) time-invariant formation (TIF, Corollary 1) for $R_i(t) = 1$ and $\boldsymbol{f}_c = (0,0)^{\mathrm{T}}$, 2) time-varying formation (TVF, Corollary 2) for $\boldsymbol{f}_c = (0,0)^{\mathrm{T}}$ and 3) time-varying formation for trajectory tracking (TVFT, Theorem 3) for $R_i(t) = 1$. The agents′ formation evolutions are shown in both the 2D plane (in Fig. 2) and 3D space (in Fig. 3). The final formation errors are shown in Fig. 4.

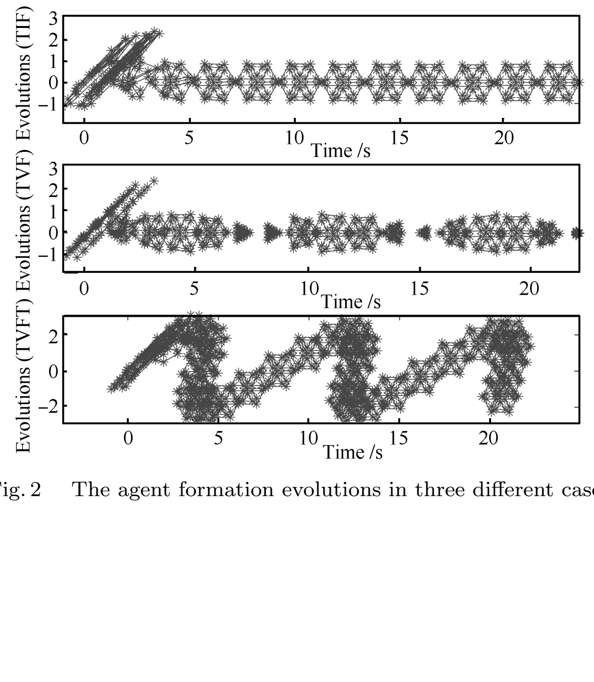

Fig. 2 The agent formation evolutions in three different cases

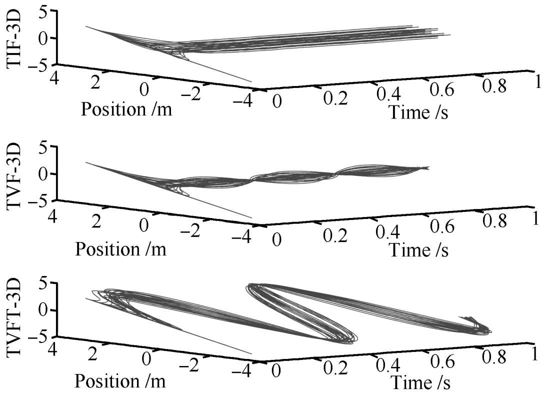

Fig. 3 The formation evolutions in 3D space

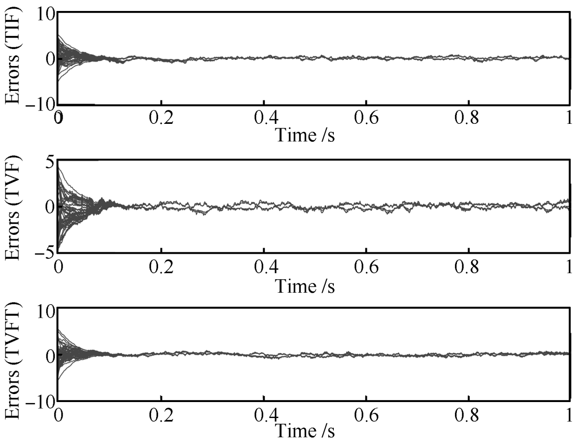

Fig. 4 The agent formation errors in three different cases

Our recent experimental results show that the proposed nonlinear formation control protocols can be widely applied to many real-world engineering systems, such as multi-machine cooperation in power grid, unmanned aerial vehicles (UAVS), autonomous underwater vehicles (AUVS), mobile robot systems (MRS) and so on.

5 Conclusion

This paper investigated the nonlinear control for multi-agent formations with directed graph topologies and time-delayed coupling in noisy environments. Control strategies based on the stability theory of stochastic delay differential equations are obtained for both time-invariant and time-varying formations as well as for time-varying formations for trajectory tracking. The results are applied to a simulated 25-agent system. It is noticed that the actual systems may be second-order, communication coupling time delay is usually time-varying and the network topology can not be always invariant. How to extend the results here to the above situation is still a challenging problem. This will be our future work.

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