Average Consensus in Directed Networks of Multi-agents with Uncertain Time-varying Delays

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Abstract This paper investigates the average consensus problem in directed networks of multi-agent systems with uncertain time-varying delays. Fixed and switching topologies that are kept weakly connected and balanced are firstly analyzed. The original system is then transformed into a reduced dimension model. Based on Jensen′s inequality and reciprocally convex approach, sufficient conditions for average consensus are further presented. Specially, a less conservative upper bound of time-varying communication delays is derived in comparison with the existing results. Numerical examples confirm the effectiveness of the proposed method.

Key words Average consensus, multi-agent systems, uncertain time-varying, reciprocally convex

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The distributed cooperative control of multi-agent systems has attracted significant amount of interest from the academics and industry due to its various advantages such as lower operational costs, less system requirements, higher robustness, and stronger adaptivity. Some recent theoretical results and progresses are mainly reviewed[1], which are classified into four directions including consensus, formation control, optimization, and estimation.

We will focus here on the first direction (i.e., consensus). Consensus and the like (synchronization, rendezvous) refer to the group behavior that all of the agents asymptotically reach a certain common agreement through a distributed protocol, which is one of the most important issues of distributed multi-agent coordination. It is ubiquitous in nature such as the fish swimming, the birds migration, the bees and ants swarm collective behavior[2]. In control engineering fields such as distributed control of multiple vehicles, and cooperation of networked multi-robots, some consensus theories and algorithms have been studied[3−7].

The consensus of multi-agent systems crucially depends on network, topology and delay, where the network may be undirected or directed graph, the topology may be fixed or switching topology, and the delay may be constant or time varying. In fact, under the real network environment, due to the finiteness of signal transmission speed, communication time delay is inevitably introduced. It is well known that time delay may degrade the system performance or even cause the system instability. Therefore, some researches have reported on the consensus analysis of multi-agent systems with time delay to investigate its effect on system performance and stability. These results can be roughly classified into two categories according to whether the system is discrete-time or continuous-time.

For discrete-time systems, the works focus on the consensus problems for low-dimensional and high-dimensional multi-agent systems with time-varying delays. The state consensus problems for low-dimensional multi-agent systems with changing communication topologies and bounded time-varying communication delays are studied[8], where only instantaneous state information of every agent can be used and some sufficient conditions for state consensus of system are presented. However, for some agents, the state consensus cannot be guaranteed generally, if only delayed information of themselves can be provided. Using not only its own instantaneous state information of every agent but also its neighbors′ instantaneous state information[9], the consensus problems for discrete-time multi-agent systems with time-varying delays and switching interaction topologies are developed. For low-dimensional multi-agent systems, some researchers have investigated the consensus problems from other aspects in recent works[10−12]. For example, a new unified framework is established to deal with the consensus in directed network of discrete-time delayed multi-agent systems with fixed topology[11]. The consensus problem for discrete-time high-dimensional linear systems with or without delays is also researched[13].

For continuous-time systems, the consensus problems for second-order multi-agent systems have been researched in the recent years[14−20], where the quantized consensus, mean square average-consensus and impulsive consensus are investigated respectively[16−18]. For the first order multi-agent systems, some researchers consider the undirected graph with fixed topology or switching topology[21−24]. Theoretical framework for posing and solving consensus problem of undirected networks with fixed topology of strongly connected and balanced digraphs with communication time-delays is presented[21], and it is shown that the maximum time-delay is inversely proportional to the largest eigenvalue of the network topology or the maximum degree of the nodes of the network. In a directed graph with fixed topology or switching topology, the average consensus problem for system with constant and time-varying delays is studied in [25], and an upper bound of time-varying communication delay is obtained. Similar results can be found in [26−29], but H_∞ consensus problems in directed networks of agents with fixed and switching topologies are investigated[26] and the recent research of noisy links with times delay is presented[28]. Moreover, the average consensus problem is considered[29], where the system with switching topology and constant time delay is only considered. The consensus problem with dynamically changing topologies and nonuniform time-varying delays is researched, where one case of intermittent communication

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and data packet dropout is also considered[30]. These researchers focus on the directed graph with fixed topology or switching topology. In recent works, some researchers have investigated the consensus problems from various perspectives[31−34]. For example, the observer-based consensus of networked multi-agent systems with time-varying delays in a sampling setting is investigated[31]. System consensus problem in three cases is analyzed[33], where the system with fixed topology contains both communication and input time delays in each case. An observer-based control strategy for networked multi-agent systems with constant communication delay and stochastic switching topology is proposed[34]. However, how to get the formulation between maximum time-delay and the network topology for a directed graph with fixed topology or switching topology is still an open issue.

In this paper, the average consensus problem for continuous-time multi-agent systems in a directed network with uncertain time-varying delays is studied. We analyze fixed and switching topologies that are kept weakly connected and balanced. Firstly, based on a reduced dimension model, a Lyapunov-Krasovskii functional with uncertain time-varying delays is constructed. Then, using Jensen′s inequality and reciprocally convex approach, sufficient conditions for average consensus are obtained by a linear matrix inequality (LMI) set, and all the agents achieve the average consensus asymptotically. The main contribution of this paper is that the average consensus of multi-agent systems in the network is presented and a less conservative upper bound of time-varying communication delay is derived in comparison with the recent results.

The paper is organized as follows. Section 1 gives some preliminaries of graph theory. The average consensus problem is formulated in Section 2. Section 3 presents the main results of the paper. Simulation results are described in Section 4, followed by conclusions in Section 5.

1 Preliminaries of graph theory

Let $G(V,E,A)$ be a directed graph of order n, where the set of nodes $V=\{v_1,\cdots,v_n\}$, the set of edges $E\subseteq V\times V$. An edge of G is denoted by $e_{ij}=(v_i,v_j)$, where v_i is the tail of the edge and v_j is the head of the edge. The set of neighbors of node v_i is denoted by $N_i=\{v_j\in V|\,(v_j,v_i)\in E\}$. The node index of G belongs to a finite index set $I=\{1,2,\cdots,n\}$. $A=[a_{ij}]$ is a weighted adjacency matrix, where the adjacency elements are positive, i.e., $a_{ij}>0$, and $a_{ii}=0$, if and only of $v_j\in N_i$, $\forall i\in I$. The in-degree and out-degree of v_i are defined as

$$d_i(v_i)=\sum_{j=1}^{n}a_{ji}\ ,\quad d_o(v_i)=\sum_{j=1}^{n}a_{ij}$$

The degree matrix $D=[d_{ij}]_{n\times n}$ is a diagonal matrix with

$$D_{ij}=\begin{cases} d_o(v_i)=\sum\limits_{j=1}^{n}a_{ij}, & i=j\\ 0, & i\neq j\end{cases}$$

The Laplacian matrix of the graph G is defined as $L=D-A$. It is noted that every row sum of L is zero and $\mathbf{1}_n=[1,1,\cdots,1]^{\mathrm{T}}\in\mathbf{R}^n$ is an eigenvector of L associated with the eigenvalue $\lambda=0$. This therefore means that $\mathrm{rank}(L)\leq n-1$.

To derive the stability criteria, some lemmas and definitions are given firstly.

Lemma 1[21]**.** If the graph G is strongly connected, then its Laplacian L satisfies:

1) $\mathrm{rank}(L)=n-1$;

2) Zero is one eigenvalue of L, and $\mathbf{1}_n$ is the corresponding eigenvector, i.e., $L\mathbf{1}_n=0$;

3) The rest $n-1$ eigenvalues all have positive real parts. In particular, if the graph G is undirected, they are all positive and real.

Definition 1 (Balanced graph[21]**).** We say the node v_i of a graph $G(V,E,A)$ is balanced if and only if its in-degree and out-degree are equal, i.e., $d_o(v_i)=d_i(v_i)$. A graph $G(V,E,A)$ is called balanced if and only if all of its nodes are balanced. Obviously, any undirected graph is balanced.

Definition 2 (Balanced matrix[25]**).** A square matrix $M\in\mathbf{R}^{n\times n}$ is said to be a balanced matrix if and only if $\mathbf{1}_n^{\mathrm{T}}M=0$ and $M\mathbf{1}_n=0$.

Lemma 2[25]**.** Consider the matrix

$$A=\begin{bmatrix} n-1 & -1 & \cdots & -1\\ -1 & n-1 & \cdots & -1\\ \vdots & \vdots & \ddots & \vdots\\ -1 & -1 & \cdots & n-1\end{bmatrix}$$

The following statements hold:

1) The eigenvalues of A are n with multiplicity $n-1$, and 0 with multiplicity 1. The vectors $\mathbf{1}_n^{\mathrm{T}}$ and $\mathbf{1}_n$ are left and the right eigenvectors of A associated with the zero eigenvalue, respectively.

2) There exists an orthogonal matrix O such that

$$O^{\mathrm{T}}AO=\begin{bmatrix} nI_{n-1} & \mathbf{0}_{(n-1)\times 1}\\ \mathbf{0}_{1\times(n-1)} & 0\end{bmatrix}$$

and O is the matrix of eigenvectors of A. For any balanced matrix $B\in\mathbf{R}^{n\times n}$,

$$O^{\mathrm{T}}BO=\begin{bmatrix} * & \mathbf{0}_{(n-1)\times 1}\\ \mathbf{0}_{1\times(n-1)} & 0\end{bmatrix}$$

Remark 1. It is obvious that when the graph G is strongly connected, its Laplacian L is a balanced matrix. According to Lemmas 1 and 2, the following equation holds:

$$U^{\mathrm{T}}LU=\begin{bmatrix} {U_1}^{\mathrm{T}}LU_1 & \mathbf{0}_{(n-1)\times 1}\\ \mathbf{0}_{1\times(n-1)} & 0\end{bmatrix}=\begin{bmatrix} \tilde{L} & \mathbf{0}_{(n-1)\times 1}\\ \mathbf{0}_{1\times(n-1)} & 0\end{bmatrix}$$

where $U=[U_1,\mathbf{1}_n/\sqrt{n}]$ is an orthogonal matrix of eigenvectors of L, and $U_1\in\mathbf{R}^{n\times(n-1)}$ is the first $n-1$ columns of U.

Lemma 3[35]**.** Let G be a balanced digraph, then G is strongly connected if and only if G is weakly connected.

Remark 2. The requirement of graph G that we discuss is strongly connected in the above, but we can obtain that it should be weakly connected by Lemma 3.

Definition 3[36]**.** Let $\Phi_1,\Phi_2,\cdots,\Phi_N:\mathbf{R}^m\mapsto\mathbf{R}^n$ be a given finite number of functions such that they have positive values in an open subset D of $\mathbf{R}^m$. Then, a reciprocally convex combination of these functions over D is a function of the form

$$\frac{1}{\alpha_1}\Phi_1+\frac{1}{\alpha_2}\Phi_2+\cdots+\frac{1}{\alpha_N}\Phi_N:D\mapsto\mathbf{R}^n$$

where the real numbers α_i satisfy $\alpha_i>0$ and $\sum_i\alpha_i=1$.

For a reciprocally convex combination of scalar positive functions $\Phi_i = f_i$, the following Lemma 4 is obtained.

Lemma 4[36]**.** Let $f_1, f_2, \cdots, f_N : \mathbf{R}^m \mapsto \mathbf{R}$ have positive values in an open subset D of $\mathbf{R}^m$. Then, the reciprocally convex combination of f_i over D satisfies

$$\min_{\{\alpha_i|\alpha_i>0,\sum_i \alpha_i=1\}} \sum_i \frac{1}{\alpha_i} f_i = \sum_i f_i + \max_{g_{i,j}(t)} \sum_{i\neq j} g_{i,j}(t)$$

s.t.

$$\left\{ g_{i,j} : \mathbf{R}^m \mapsto \mathbf{R}, g_{j,i}(t) \triangleq g_{i,j}(t), \begin{bmatrix} f_i(t) & g_{i,j}(t) \\ g_{j,i}(t) & f_j(t) \end{bmatrix} \geq 0 \right\}$$

2 Consensus protocol formulation

Suppose that the network system consists of n agents. Each agent is regarded as a node in a directed graph G, and $x_i(t) \in \mathbf{R}$ represents the state of the i-th node. Moreover, suppose each node is a dynamic agent with dynamics:

$$\dot{x}_i(t) = u_i(t)$$

where $u_i(t)$ is the control input (or protocol), $i \in I = \{1, 2, \cdots, n\}$.

We say the nodes of a network achieve a consensus if and only if $x_i = x_j$ for all $i \neq j$, $i, j \in I$. Particularly, if

$$\lim_{t\to\infty} x_i(t) = \frac{1}{n} \sum_j x_j(0)$$

we say protocol u_i asymptotically solves the average-consensus problem.

For a fixed topology without communication time-delay, the consensus protocol is used[21]:

$$u_i(t) = \sum_{v_j \in N_i} a_{ij}(t)(x_j(t) - x_i(t))$$

In this paper, we discuss the average-consensus problem in networks of agents with time-varying delays. The agreement control law[21] is given by

$$u_i(t) = \sum_{v_j \in N_i} a_{ij}(t)(x_j(t - \tau(t)) - x_i(t - \tau(t))) \tag{1}$$

where $\tau(t)$ is a time-varying delay.

Then multi-agent systems with fixed topology G and uncertain time-varying communication delays is formulated by

$$\dot{x}(t) = -Lx(t - \tau(t)) \tag{2}$$

where $x(t) = [x_1(t), x_2(t), \cdots, x_n(t)]^{\mathrm{T}}$ denotes the value of all nodes, and L is the Laplacian of graph G.

Consider the following multi-agent systems with switching topology G_k and uncertain time-varying communication delays

$$\dot{x}(t) = -L_k x(t - \tau_k(t)), k = s(t) \tag{3}$$

where L_k is the Laplacian of graph $G_k (G_k \in \Gamma_n)$, $\Gamma_n = \{G = (V, E, A)\}$ is a finite collection of digraphs of order n that is balanced and weakly connected, $s(t) : [0, +\infty] \to T_{\Gamma_n} = \{1, 2, \cdots, N\}$ ($N \in \mathbf{Z}^+$ denotes the total number of all possible directed graphs) is a switching signal that determines the communication topology, and $\tau_k(t)$ satisfies $0 \leq \tau_{k1} \leq \tau_k(t) \leq \tau_{k2}$, where τ_{k1} and τ_{k2} denote the lower and upper delay bounds to be determined in this paper. Moreover, the initial conditions of (3) is assumed to satisfy $\phi(t) \equiv x(0) = [x_1(0), \cdots, x_n(0)]^{\mathrm{T}}, -\tau_{k\max} \leq t \leq 0$, with $\tau_{k\max} = \max\{\tau_{k2}\}$.

Remark 3. Considering a hybrid system with a continuous-state $x \in \mathbf{R}^n$ and a discrete-state $G_k \in {\Gamma_n}$[21], the average consensus problem in a directed network of multi-agent systems with switching topology and time-varying communication delays has been investigated. When any digraph G_k is strongly or weakly connected and balanced, a common Lyapunov-Krasovskii functional can be found to solve the system average consensus problem with uncertain time-varying delays. For each time-changing L_k, there exists a maximum allowable delay $\tau_{k\max}$. To satisfy for all the possible communication topologies G_k, the maximum allowable delay should be $\tau_{\max} = \min\{\tau_{k\max}\}$.

3 Main results

Theorem 1. For system (3), average consensus can be achieved if there exist positive definite matrices $\tilde{P}, \tilde{Q}_1, \tilde{Q}_2, \tilde{R}_1, \tilde{R}_2 \in \mathbf{R}^{(n-1)\times(n-1)}$ and matrix $\tilde{S}_2$ with $(n-1)$ dimensions, such that

$$\begin{bmatrix} \tilde{R}_2 & \tilde{S}_2^{\mathrm{T}} \\ \tilde{S}_2 & \tilde{R}_2 \end{bmatrix} \geq 0 \tag{4}$$

$$\begin{bmatrix} \Pi_{11} & * & * & * & * & * \\ \Pi_{21} & \Pi_{22} & * & * & * & * \\ \Pi_{31} & \Pi_{32} & \Pi_{33} & * & * & * \\ 0 & \Pi_{42} & \Pi_{43} & \Pi_{44} & * & * \\ 0 & \Pi_{52} & 0 & 0 & \Pi_{55} & * \\ 0 & \Pi_{62} & 0 & 0 & 0 & \Pi_{66} \end{bmatrix} < 0 \tag{5}$$

where $\Pi_{11} = \tilde{Q}_1 + \tilde{Q}_2 - \tilde{R}_1$, $\Pi_{21} = -\tilde{L}_k^{\mathrm{T}} \tilde{P}$, $\Pi_{22} = -2\tilde{R}_2 + \tilde{S}_2 + \tilde{S}_2^{\mathrm{T}}$, $\Pi_{31} = \tilde{R}_1$, $\Pi_{32} = \tilde{R}_2 - \tilde{S}_2^{\mathrm{T}}$, $\Pi_{33} = -\tilde{Q}_1 - \tilde{R}_1 - \tilde{R}_2$, $\Pi_{42} = \tilde{R}_2 - \tilde{S}_2$, $\Pi_{43} = \tilde{S}_2$, $\Pi_{44} = -\tilde{Q}_2 - \tilde{R}_2$, $\Pi_{52} = -\tau_{k1}\tilde{R}_1 \tilde{L}$, $\Pi_{55} = -\tilde{R}_1$, $\Pi_{62} = (\tau_{k2} - \tau_{k1})\tilde{R}_2 \tilde{L}_k$, $\Pi_{66} = -\tilde{R}_2$, $\tilde{L}_k = U_1^{\mathrm{T}} L_k U_1$, $U_1 \in \mathbf{R}^{n\times(n-1)}$ is the first $n-1$ columns of U, and U is an orthogonal matrix of eigenvectors of L_k.

Proof. According to the directed balanced graph with switching topology, we have $\mathbf{1}_n^{\mathrm{T}} L_k = 0$. It indicates that

$$\sum_i \dot{x}_i(t) = \sum_i u_i(t) = 0$$

Further, it is known that $\alpha = Ave(x(t))$ is an invariant quantity, and $x(t)$ can be decomposed as[21]

$$x(t) = \alpha \mathbf{1}_n + \delta(t) \tag{6}$$

where $\delta(t) \in \mathbf{R}^n$ satisfies $\sum_i \delta_i(t) = 0$.

Next, using (6), system (3) can be re-written as

$$\begin{aligned} \dot{\delta}(t) &= -L_k \delta(t - \tau_k(t)), \\ & 0 \leq \tau_{k1} \leq \tau_k(t) \leq \tau_{k2} \end{aligned} \tag{7}$$

According to Remark 1, (7) becomes

$$U^{\mathrm{T}} \dot{\delta}(t) = -U^{\mathrm{T}} L_k U U^{\mathrm{T}} \delta(t - \tau_k(t)) = \\ -\begin{bmatrix} \tilde{L}_k & \mathbf{0}_{(n-1)\times 1} \\ \mathbf{0}_{1\times(n-1)} & 0 \end{bmatrix} U^{\mathrm{T}} \delta(t - \tau_k(t))$$

Considering that $\sum_i \delta_i(t - \tau_k(t)) = 0, \sum_i \dot{\delta}_i(t) = 0$, we have

$$U^{\mathrm{T}}\dot{\delta}(t) = [*,\cdots,*,\ 0]^{\mathrm{T}} = [\dot{\tilde{\delta}}^{\mathrm{T}}(t), 0]^{\mathrm{T}}$$
$$U^{\mathrm{T}}\delta(t-\tau_k(t)) = [\tilde{\delta}^{\mathrm{T}}(t-\tau_k(t), 0]^{\mathrm{T}}$$

Therefore, system (7) is equivalent to

$$\begin{aligned}\dot{\tilde{\delta}}(t) = -\tilde{L}_k\tilde{\delta}(t-\tau_k(t)),\\ 0\le\tau_{k1}\le\tau_k(t)\le\tau_{k2}\end{aligned}\tag{8}$$

where the vector $\tilde{\delta}$ is an $(n-1)$-dimensional subspace called the disagreement eigenspace of L_k, $\tilde{L}_k\in\mathbf{R}^{(n-1)\times(n-1)}$ and $\mathrm{rank}(\tilde{L}_k) = n-1$.

Then, it is known that if $\lim\limits_{t\to\infty}\|\tilde{\delta}(t)\| = 0$, then $\lim\limits_{t\to\infty}\|\delta(t)\| = 0$[27].

Considering system (8), a Lyapunov-Krasovskii function is constructed as

$$\begin{aligned}V(t) =\ & \tilde{\delta}^{\mathrm{T}}(t)\tilde{P}\tilde{\delta}(t)+\\ & \int_{t-\tau_{k1}}^{t}\tilde{\delta}^{\mathrm{T}}(\alpha)\tilde{Q}_1\tilde{\delta}(\alpha)\mathrm{d}\alpha+\\ & \int_{t-\tau_{k2}}^{t}\tilde{\delta}^{\mathrm{T}}(\alpha)\tilde{Q}_2\tilde{\delta}(\alpha)\mathrm{d}\alpha+\\ & \tau_{k1}\int_{-\tau_{k1}}^{0}\int_{t+\alpha}^{t}\dot{\tilde{\delta}}^{\mathrm{T}}(\beta)\tilde{R}_1\dot{\tilde{\delta}}(\beta)\mathrm{d}\beta\mathrm{d}\alpha+\\ & (\tau_{k2}-\tau_{k1})\int_{-\tau_{k2}}^{-\tau_{k1}}\int_{t+\alpha}^{t}\dot{\tilde{\delta}}^{\mathrm{T}}(\beta)\tilde{R}_2\dot{\tilde{\delta}}(\beta)\mathrm{d}\beta\mathrm{d}\alpha\end{aligned}\tag{9}$$

where $\tilde{P}, \tilde{Q}_1, \tilde{Q}_2, \tilde{R}_1, \tilde{R}_2\in\mathbf{R}^{(n-1)\times(n-1)}$ are positive definite matrices and $\tilde{S}_2\in\mathbf{R}^{(n-1)\times(n-1)}$.

The derivative of (9) with respect to t is

$$\begin{aligned}\dot{V}(t) =\ & 2\tilde{\delta}^{\mathrm{T}}(t)\tilde{P}\dot{\tilde{\delta}}(t)+\tilde{\delta}^{\mathrm{T}}(t)\tilde{Q}_1\tilde{\delta}(t)-\\ & \tilde{\delta}^{\mathrm{T}}(t-\tau_{k1})\tilde{Q}_1\tilde{\delta}(t-\tau_{k1})+\tilde{\delta}^{\mathrm{T}}(t)\tilde{Q}_2\tilde{\delta}(t)-\\ & \tilde{\delta}^{\mathrm{T}}(t-\tau_{k2})\tilde{Q}_2\tilde{\delta}(t-\tau_{k2})+\tau_{k1}^2\dot{\tilde{\delta}}^{\mathrm{T}}(t)\tilde{R}_1\dot{\tilde{\delta}}(t)-\\ & \tau_{k1}\int_{t-\tau_{k1}}^{t}\dot{\tilde{\delta}}^{\mathrm{T}}(\alpha)\tilde{R}_1\dot{\tilde{\delta}}(\alpha)\mathrm{d}\alpha+\\ & (\tau_{k2}-\tau_{k1})^2\dot{\tilde{\delta}}^{\mathrm{T}}(t)\tilde{R}_2\dot{\tilde{\delta}}(t)-\\ & (\tau_{k2}-\tau_{k1})\int_{t-\tau_{k2}}^{t-\tau_{k1}}\dot{\tilde{\delta}}^{\mathrm{T}}(\alpha)\tilde{R}_2\dot{\tilde{\delta}}(\alpha)\mathrm{d}\alpha\end{aligned}\tag{10}$$

Define

$$\chi(t) := [\tilde{\delta}^{\mathrm{T}}(t),\ \tilde{\delta}^{\mathrm{T}}(t-\tau_k(t)),\ \tilde{\delta}^{\mathrm{T}}(t-\tau_{k1}),\ \tilde{\delta}^{\mathrm{T}}(t-\tau_{k2})]^{\mathrm{T}}$$

$e_1 := \begin{bmatrix} I & 0 & 0 & 0\end{bmatrix}^{\mathrm{T}}$, $e_2 := \begin{bmatrix} 0 & I & 0 & 0\end{bmatrix}^{\mathrm{T}}$, $e_3 := \begin{bmatrix} 0 & 0 & I & 0\end{bmatrix}^{\mathrm{T}}$, $e_4 := \begin{bmatrix} 0 & 0 & 0 & I\end{bmatrix}^{\mathrm{T}}$ and $e_5 := (-\tilde{L}_k{e_2}^{\mathrm{T}})^{\mathrm{T}}$, where $I\in\mathbf{R}^{(n-1)\times(n-1)}$ is an unit matrix and $0\in\mathbf{R}^{(n-1)\times(n-1)}$ is a zero matrix..

To handle the integral items in (10), using Jensen′s inequality[37] yields,

$$\begin{aligned}&\tau_{k1}\int_{t-\tau_{k1}}^{t}\dot{\tilde{\delta}}^{\mathrm{T}}(\alpha)\tilde{R}_1\dot{\tilde{\delta}}(\alpha)\mathrm{d}\alpha\ge\\ &\quad\tau_{k1}\times\frac{1}{\tau_{k1}}\sum_{l=1}^{n}\dot{\tilde{\delta}}^{\mathrm{T}}(\varepsilon)\Delta t_i\tilde{R}_1\sum_{l=1}^{n}\dot{\tilde{\delta}}(\varepsilon)\Delta t_i =\\ &\quad\left(\tilde{\delta}(t)-\tilde{\delta}(t-\tau_{k1})\right)^{\mathrm{T}}\tilde{R}_1\left(\tilde{\delta}(t)-\tilde{\delta}(t-\tau_{k1})\right) =\\ &\quad\chi^{\mathrm{T}}(t)(e_1-e_3)\tilde{R}_1(e_1-e_3)^{\mathrm{T}}\chi(t)\end{aligned}\tag{11}$$

$$\begin{aligned}&(\tau_{k2}-\tau_{k1})\int_{t-\tau_{k2}}^{t-\tau_{k1}}\dot{\tilde{\delta}}^{\mathrm{T}}(\alpha)\tilde{R}_2\dot{\tilde{\delta}}(\alpha)\mathrm{d}\alpha =\\ &\quad(\tau_{k2}-\tau_{k1})\int_{t-\tau_k(t)}^{t-\tau_{k1}}\dot{\tilde{\delta}}^{\mathrm{T}}(\alpha)\tilde{R}_2\dot{\tilde{\delta}}(\alpha)\mathrm{d}\alpha+\\ &\quad(\tau_{k2}-\tau_{k1})\int_{t-\tau_{k2}}^{t-\tau_k(t)}\dot{\tilde{\delta}}^{\mathrm{T}}(\alpha)\tilde{R}_2\dot{\tilde{\delta}}(\alpha)\mathrm{d}\alpha\ge\\ &\quad\frac{(\tau_{k2}-\tau_{k1})}{\tau_k(t)-\tau_{k1}}\chi^{\mathrm{T}}(t)(e_3-e_2)\tilde{R}_2(e_3-e_2)^{\mathrm{T}}\chi(t)+\\ &\quad\frac{(\tau_{k2}-\tau_{k1})}{\tau_{k2}-\tau_k(t)}\chi^{\mathrm{T}}(t)(e_2-e_4)\tilde{R}_2(e_2-e_4)^{\mathrm{T}}\chi(t)\end{aligned}\tag{12}$$

Using Lemma 4 to (12) yields

$$(\tau_{k2}-\tau_{k1})\int_{t-\tau_{k2}}^{t-\tau_{k1}}\dot{\tilde{\delta}}^{\mathrm{T}}(\alpha)\tilde{R}_2\dot{\tilde{\delta}}(\alpha)\mathrm{d}\alpha\ge\chi(t)^{\mathrm{T}}\Lambda_1\chi(t)\tag{13}$$

where $\Lambda_1 = \begin{bmatrix}(e_3-e_2)^{\mathrm{T}}\\ (e_2-e_4)^{\mathrm{T}}\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}\tilde{R}_2 & \tilde{S}_2^{\mathrm{T}}\\ \tilde{S}_2 & \tilde{R}_2\end{bmatrix}\begin{bmatrix}(e_3-e_2)^{\mathrm{T}}\\ (e_2-e_4)^{\mathrm{T}}\end{bmatrix}$, and $\begin{bmatrix}\tilde{R}_2 & \tilde{S}_2^{\mathrm{T}}\\ \tilde{S}_2 & \tilde{R}_2\end{bmatrix}\ge 0$.

Substituting (11), (13) into (10) yields

$$\begin{aligned}\dot{V}(t)\le\ &\chi^{\mathrm{T}}[e_5\tilde{P}e_1^{\mathrm{T}}+e_1\tilde{P}e_5^{\mathrm{T}}+e_1\tilde{Q}_1e_1^{\mathrm{T}}-e_3\tilde{Q}_1e_3^{\mathrm{T}}+e_1\tilde{Q}_2e_1^{\mathrm{T}}-\\ & e_4\tilde{Q}_2e_4^{\mathrm{T}}+\tau_{k1}^2e_5\tilde{R}_1e_5^{\mathrm{T}}+(\tau_{k2}-\tau_{k1})^2e_5\tilde{R}_2e_5^{\mathrm{T}}-\\ & (e_1-e_3)\tilde{R}_1(e_1-e_3)^{\mathrm{T}}-\Lambda_1\,]\chi(t)\end{aligned}$$

According to Lyapunov stability theory, if $\dot{V}(t)<0$, the system (8) is asymptotically stable. So we have

$$\begin{aligned}& e_5\tilde{P}e_1^{\mathrm{T}}+e_1\tilde{P}e_5^{\mathrm{T}}+e_1\tilde{Q}_1e_1^{\mathrm{T}}-e_3\tilde{Q}_1e_3^{\mathrm{T}}+e_1\tilde{Q}_2e_1^{\mathrm{T}}-\\ &\quad e_4\tilde{Q}_2e_4^{\mathrm{T}}+\tau_{k1}^2e_5\tilde{R}_1e_5^{\mathrm{T}}+(\tau_{k2}-\tau_{k1})^2e_5\tilde{R}_2e_5^{\mathrm{T}}-\\ &\quad(e_1-e_3)\tilde{R}_1(e_1-e_3)^{\mathrm{T}}-\\ &\quad\begin{bmatrix}(e_3-e_2)^{\mathrm{T}}\\ (e_2-e_4)^{\mathrm{T}}\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}\tilde{R}_2 & \tilde{S}_2^{\mathrm{T}}\\ \tilde{S}_2 & \tilde{R}_2\end{bmatrix}\begin{bmatrix}(e_3-e_2)^{\mathrm{T}}\\ (e_2-e_4)^{\mathrm{T}}\end{bmatrix} =\\ &\quad\begin{bmatrix}\Pi_{11} & * & * & *\\ \Pi_{21} & \Pi_{22} & * & *\\ \Pi_{31} & \Pi_{32} & \Pi_{33} & *\\ 0 & \Pi_{42} & \Pi_{43} & \Pi_{44}\end{bmatrix}+\\ &\quad\tau_{k1}^2e_5\tilde{R}_1e_5^{\mathrm{T}}+(\tau_{k2}-\tau_{k1})^2e_5\tilde{R}_2e_5^{\mathrm{T}}<0\end{aligned}\tag{14}$$

Finally, using Schur complement, (14) is equivalent to (5). □

Remark 4. Since switching topology is employed in system (3), i.e., L_k is time-varying, the LMI (5) should hold for all the possible communication topologies.

Remark 5. In Theorem 1, Jensen′s inequality and reciprocally convex approach are used to deal with the double integral terms of the Lyapunov-Krasovskii functional in (9), which significantly reduces the conservativeness of the stability conditions in comparison with the methods in [25, 27]. When the switching topology that is kept strongly or weakly connected and balanced, a less conservative upper bound of time-varying communication delays can be obtained by using Theorem 1, thus the proposed method is more general. This will lay a solid foundation for the further development of multi-agent systems with time-delay.

Remark 6. It is well known that any undirected graph is balanced. Therefore, the proposed method can be applied to undirected topology with communication time-varying delays. Moreover, the proposed method can be extended to continuous-time and discrete-time dynamic networks with fixed topology, which is balanced and weakly connected.

How to get the maximum delay bounds is given as follow:

Step 1. Set $\tau_{k1} = \tau_0$, $\tau_{k2} = \tau_0$ and step size $h = h_0$, where τ_0 and h_0 are specified constants and small enough.

Step 2. Based on LMI approach, the feasible solutions that satisfy the matrix inequalities (4) and (5) are searched. If the feasible solutions can be found, then set $\tau_{k2} = \tau_{k2} + h$ and return to Step 2. Otherwise, stop and τ_{k2} is the maximum delay bound.

The following Theorem 2 gives sufficient conditions in terms of LMIs for system (2) with fixed topology.

Theorem 2. Consider a directed network of agents with uncertain time-varying communication delays $\tau(t)$ satisfying $0 \le \tau_1 \le \tau(t) \le \tau_2$, where τ_1 and τ_2 are constants to be determined. Assume the network has a balanced and weakly connected graph. Then, system (2) asymptotically solves average consensus problem if there exist positive definite matrixes $\tilde{P}, \tilde{Q}_1, \tilde{Q}_2, \tilde{R}_1, \tilde{R}_2 \in \mathbf{R}^{(n-1)\times(n-1)}$ and matrix $\tilde{S}_2$ with $(n-1)$ dimensions, such that

$$\begin{bmatrix} \tilde{R}_2 & \tilde{S}_2^{\mathrm{T}} \\ \tilde{S}_2 & \tilde{R}_2 \end{bmatrix} \ge 0 \qquad (15)$$

$$\begin{bmatrix} \Pi_{11} & * & * & * & * & * \\ \Pi_{21} & \Pi_{22} & * & * & * & * \\ \Pi_{31} & \Pi_{32} & \Pi_{33} & * & * & * \\ 0 & \Pi_{42} & \Pi_{43} & \Pi_{44} & * & * \\ 0 & \Pi_{52} & 0 & 0 & \Pi_{55} & * \\ 0 & \Pi_{62} & 0 & 0 & 0 & \Pi_{66} \end{bmatrix} < 0 \qquad (16)$$

where $\Pi_{11} = \tilde{Q}_1 + \tilde{Q}_2 - \tilde{R}_1$, $\Pi_{21} = -\tilde{L}^{\mathrm{T}}\tilde{P}$, $\Pi_{22} = -2\tilde{R}_2 + \tilde{S}_2 + \tilde{S}_2^{\mathrm{T}}$, $\Pi_{31} = \tilde{R}_1$, $\Pi_{32} = \tilde{R}_2 - \tilde{S}_2^{\mathrm{T}}$, $\Pi_{33} = -\tilde{Q}_1 - \tilde{R}_1 - \tilde{R}_2$, $\Pi_{42} = \tilde{R}_2 - \tilde{S}_2$, $\Pi_{43} = \tilde{S}_2$, $\Pi_{44} = -\tilde{Q}_2 - \tilde{R}_2$, $\Pi_{52} = -\tau_1\tilde{R}_1\tilde{L}$, $\Pi_{55} = -\tilde{R}_1$, $\Pi_{62} = (\tau_2 - \tau_1)\tilde{R}_2\tilde{L}$, $\Pi_{66} = -\tilde{R}_2$, $\tilde{L} = U_1^{\mathrm{T}} L U_1$, $U_1 \in \mathbf{R}^{n\times(n-1)}$ is the first $n-1$ columns of U, and U is an orthogonal matrix of eigenvectors of L.

Proof. The network topology G is balanced and weakly connected, and $x(t)$ can be decomposed as[21] $x(t) = \alpha\mathbf{1}_n + \delta(t)$. Then system (2) is equivalent to the following $n-1$ dimensional system:

$$\dot{\delta}(t) = -L\delta(t-\tau(t)),\ 0 \le \tau_1 \le \tau(t) \le \tau_2$$

The Lyapunov function $V(t)$ in (9) does not depend on the network topology. Thus, $V(t)$ is also a valid Lyapunov function for the stability analysis of the system (2). The following proof is similar to that of Theorem 1 and omitted here. □

Similarly, how to get the maximum delay bounds is also given:

Step 1. Set $\tau_1 = \tau_0$, $\tau_2 = \tau_0$ and step size $h = h_0$, where τ_0 and h_0 are specified constants and small enough.

Step 2. Based on LMI approach, the feasible solutions that satisfy the matrix inequalities (15) and (16) are searched. If the feasible solutions can be found, then set $\tau_2 = \tau_2 + h$ and return to Step 2. Otherwise, stop and τ_2 is the maximum delay bound.

4 Numerical examples

To verify the effectiveness of proposed method, three numerical examples were operated.

Example 1. Consider a directed switching network of ten agents[27] as shown in Fig. 1, where all digraphs have 0-1 weights. Fig. 2 shows a finite automation state machine with four states G_1, G_2, G_3, G_4. Further, suppose that the switching signal can switch arbitrarily fast in the four possible topologies. It is seen obviously from Fig. 1 that they are all balanced and weakly connected. Set the initial condition $\alpha = \mathrm{Ave}(x(0)) = \frac{1}{n}\sum_i x_i(0) = 0$.

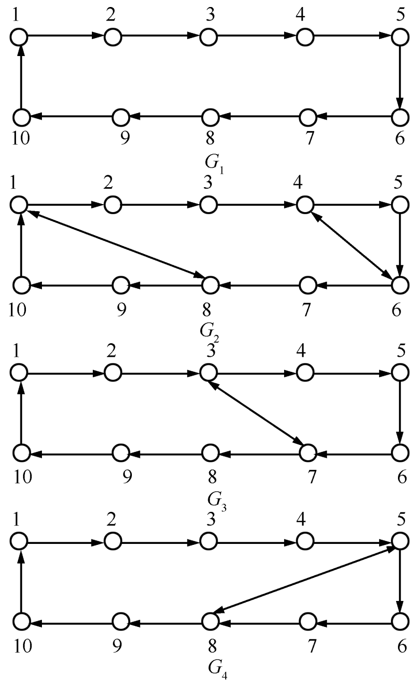

Fig. 1 Four directed graphs

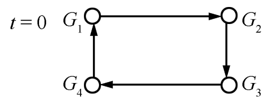

Fig. 2 A finite automation state machine

The maximum allowable delay $\tau_{\max}$ as shown in Table 1 is derived according to Theorem 1. Further, the maximum allowable delay of other two methods in [25, 27] are also listed in Table 1. It is found that the proposed method is less conservative than other two methods. Fig. 3 shows the state trajectories of the above network system with $\tau_k = 0.38\,\mathrm{s}$. It is seen obviously that average consensus is asymptotically achieved.

Table 1 Comparison of maximum allowable delays

Methods	$\tau_{\max}$
The method in [25]	0.30
The method in [27]	0.31
The proposed method	0.40

Example 2. Consider a directed graph of six agents[30] as shown in Fig. 4. Topology G has balanced and weakly connected digraph with 0-1 weights. Set the initial condition $\alpha = \text{Ave}(x(0)) = 0$.

For fixed topology G in Fig. 4, using Theorem 2 of the presented method to solve average consensus problem, the maximum allowable delay is 0.517. Fig. 5 shows the state trajectories of the topology G. Using the method in [30], the allowable upper bound of delay is 0.499 for topology G, it has shown that the proposed method reduces the conservativeness of maximum allowable delay. It confirms the effectiveness of the proposed method again.

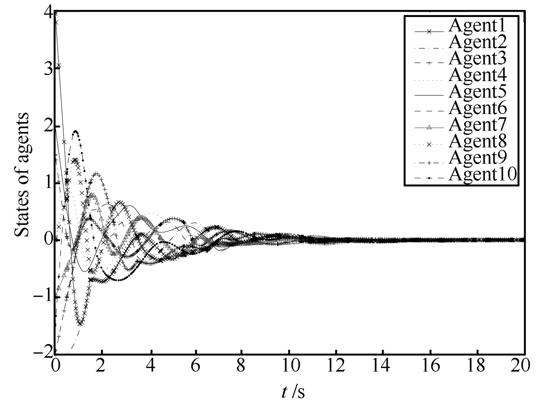

Fig. 3 State trajectories of the network with switching topology and $\tau_k = 0.38$

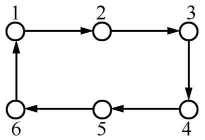

Fig. 4 A directed graph

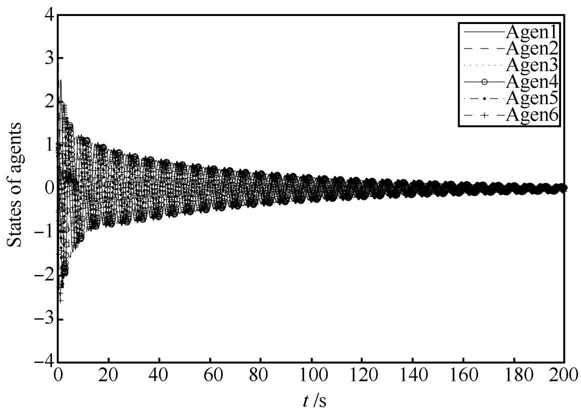

Fig. 5 State trajectories of the topology G with time-delay $\tau_2 = 0.505$

Remark 7. For a directed graph, the case with uncertain time-varying delays and fixed topology has been discussed[30] based on a tree-type transformation approach. However, from the simulation results of Example 2, the proposed method is more effective. Therefore, the Jensen′s inequality and reciprocally convex approach play a key role in the asymptotic stability analysis of the consensus protocol in this paper.

Example 3. An undirected network of six agents is shown in Fig. 6, where the digraph has 0-1 weights. Using Theorem 2 of the proposed method to solve the average consensus problem, it can be obtained that the maximum delay bound is 0.2507. Using the method in [21], this system can achieve average consensus asymptotically if and only if $\tau_{\max} \leq \pi/2\lambda_{\max}(L) = 0.2936$, where $\lambda_{\max}(L)$ is the maximum eigenvalue of Laplacian of the undirected graph. The maximum delay bound based on the method in [25] is 0.1871. It can be shown that the maximum delay bound by Theorem 2 is very close to its critical value. Thus the proposed method is suitable for the undirected cases.

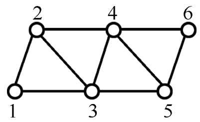

Fig. 6 One undirected graph

5 Conclusions

This paper has mainly investigated the average consensus problem in a directed network of multi-agent systems with fixed/switching topology as well as uncertain time-varying communication delays. Sufficient conditions for average consensus are presented, and a less conservative upper bound of time-varying communication delays is derived. In comparison with the existing methods[25, 27, 30], the obtained results of average consensus problem reduce the conservativeness of the stability conditions, and the switching topology is no longer required strongly connected in a directed network, thus the obtained results are more general. However, due to unreliable information channels and limited bandwidth, communication between agents may produce data packet dropout and out-of-order. Therefore, considering these network-related non-deterministic issues, how to study the average consensus problem is the further work. Moreover, how to save the bandwidth by the triggered mechanism is another important direction in the future.

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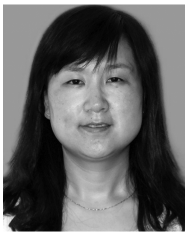

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