Partial Stability Approach to Consensus Problem of Linear Multi-agent Systems

CHEN Yang-Zhou[1] GE Yan-Rong[1, 2] ZHANG Ya-Xiao[1]

Abstract A linear transformation is proposed to deal with the consensus problem of high-order linear multi-agent systems (LMASs). In virtue of the linear transformation, the consensus problem is equivalently translated into a partial stability problem. We discuss three issues of the LMASs under a generalized linear protocol: 1) to find criteria of consensus convergence; 2) to calculate consensus function; 3) to design gain matrices in the linear consensus protocol. Precisely, we provide a necessary and sufficient criterion of consensus convergence in terms of Hurwitz stability of a matrix and give an analytical expression of the consensus function. In addition, we set up a relation between the gain matrices in the protocol and the convergence time and consensus accuracy of the agents, and then design the gain matrices with respect to a pre-specified convergence time and a required consensus accuracy.

Key words Multi-agent system, criterion of consensus convergence, consensus function/value, linear transformation, partial stability

Citation Chen Yang-Zhou, Ge Yan-Rong, Zhang Ya-Xiao. Partial stability approach to consensus problem of linear multi-agent systems. *Acta Automatica Sinica*, 2014, **40**(11): 2573−2584

DOI 10.3724/SP.J.1004.2014.02573

In recent years, the consensus problem is well recognized as a fundamental problem in the cooperation control of multi-agent systems and thus has caught a lot of attention of the researchers from various communities. The interest in this problem is motivated by a huge variety of applications, e.g., work load balance in parallel computer networks[1], coverage control[2], clock synchronization[3], consensus filtering and estimation in sensor networks[4−5], rendezvous and formation of various moving objects such as underwater vehicles[6], aircraft[7−8], satellites[9], mobile robots[10], intelligent vehicles in automated highway systems[11], etc. For a nice overview of recent results on these topics, please refer to [12−14] and the references therein.

A multi-agent system is composed of multiple interacting dynamic objects. To model this class of systems, one needs to consider three essential elements: 1) a model describing dynamics of the agents; 2) a communication topology (graph or network) describing communication structure between the agents; 3) a protocol (control input) describing how the agents interact with each other according to a given communication topology. The dynamical models describing states (some information variables) of the agents are often represented by differential or difference equations. These models are distinguished as various cases, such as continuous-time or discrete-time, linear or nonlinear, low-order or high-order, certain or uncertain, etc. The communication topologies also include such cases as undirected or directed, fixed or switched and so on. The protocols of the agents are constructed depending on the states of the agent itself and its "neighboring" agents and thus they are typically local or decentralized controls. The protocols are classified as such cases as linear or nonlinear, in absence or presence of communication delay and so on.

Roughly speaking, solving the consensus problem of a multi-agent system is to construct a communication topology and a protocol with respect to the communication topology such that all the agents achieve a common state of the interesting variables. According to the requirements of the common state, the consensus problems are sorted into as leaderless or leader-following consensus, state or output consensus and so on.

Different combinations of aforementioned cases result in various complex settings of the consensus problem. Lots of researches have been done and abundant results have been obtained. However, the aim of this paper consists in trying to provide a novel uniform approach for the study of the consensus problem. In order to give prominence to our idea, we do not entangle ourself with the complex settings but rather backtrack to one of the simplest cases, i.e., the continuous-time, leaderless and certain linear multi-agent systems with fixed communication topology and in absence of communication delay. Hence, below we will limit ourself to summarize some results only on this simple case.

The initial research on the simple case was dedicated to the protocol construction and the convergence criterion of the first-order or single-integrator linear leaderless systems. Olfati-Saber et al.[15] first realized the importance of the relation between the consensus problem and the graph Laplacian expressing the communication topology, where all the gains in a linear protocol were set to be 1. Then the authors in [16−18] showed that the agents achieve global consensus if and only if the associated graph has a directed spanning tree. Average consensus was defined in [19] and it was shown that average consensus is achieved if the graph is both strongly connected and balanced. Wu et al.[18] further proved that if the graph is strongly connected but not balanced, the average consensus problem is still solved by using multi-rate integrators.

The next study was extended to the second-order or double-integrator linear leaderless systems, which was nontrivial as shown in the literature. Ren et al.[20] proposed a protocol consisting of both weighted position and velocity differences between the agent itself and its "neighbors" and proved that the directed spanning tree is still necessary for the consensus convergence but not sufficient any more except adding a condition on the velocity weights. Xie et al.[21] proposed another protocol consisting of velocity feedback of the agent itself and weighted position difference between the agent and its "neighbors" and proved that, if the

Manuscript received June 19, 2013; accepted November 21, 2013
Supported by National Natural Science Foundation of China (61079001, 61273006), National High Technology Research and Development Program of China (863 Program) (2011AA110301), and Specialized Research Fund for the Doctoral Program of Higher Education of China (20111103110017)
Recommended by Associate Editor CHEN Jie
1. College of Electronic Information and Control Engineering, Beijing University of Technology, Beijing 100124, China 2. College of Physics Science and Information Engineering, Hebei Normal University, Shijiazhuang 050024, China

graph is connected and the initial velocity of each agent is zero, the agents can achieve global average-consensus for any negative gain in the velocity feedback. More cases and corresponding consensus protocols on the double-integrator system were discussed in [22]. Yu et al.[23] gave a necessary and sufficient consensus criterion.

The extension from low order systems to high order systems is not a trivial issue. de Castro et al.[24] provided necessary and sufficient consensus conditions for high-order linear multi-agents via linear matrix inequality. Ren et al.[25] studied a special high-order model, which can be regarded as a special controllability canonical form. Wang et al.[26] considered a high-order model with fewer structural limitations and presented a sufficient condition for consensus convergence under the assumption of undirected graph. Xiao et al.[27] studied a general high-order model and provided some necessary and sufficient criteria. Munz et al.[28] proposed a high order differential equation model of the agents and provided a Nyquist consensus criterion. Xi et al.[29] proposed a space decomposition method to study the high order systems.

More complex settings of the consensus problem have been researched, including the case with time-delays in the protocol[30−44], which may arise due to congestion of communication channels, asymmetry of interactions, finite transmission speed, etc., and the case with the switching information topology[19−20, 32−33, 39, 42, 45], which often appears due to various factors such as disturbances or communication range limit. The other interesting topics are concerned with consensus convergence speed[19, 46−49], finite-time consensus problems[4, 50−52], consensus of nonlinear systems[53], consensus of discrete-time systems[54−55], sampled-data based average consensus[56] and the parallel-developing leader-following consensus problem. We omit many (maybe important) papers and detailed reviews of these cases because of our research limitation of the simple case mentioned above.

In the paper, we consider the linear multi-agent systems (LMASs) in a general form, but limit the setting of consensus problem to the simplest case mentioned above. For this simplest case, we propose a generalized linear protocol and address three aspects of the problem: 1) to find criteria of consensus convergence; 2) to calculate the consensus function; 3) to design the gain matrices in the consensus protocol.

Our contributions are summarized as follows. Firstly, we propose a proper linear transformation to equivalently transform the consensus problem into a stability problem with respect to partial variables (partial stability for short) of the transformed system. Secondly, we derive a necessary and sufficient criterion of consensus convergence in terms of matrix Hurwitz stability and give an analytical expression of the consensus function. Furthermore, applying the general result to the single/double-integrator linear systems, respectively, we get easier testing necessary and sufficient criteria and give analytical formulae of the consensus values/functions as well. Applying the results to the average consensus problem, we improve the existing results as well. Thirdly, for the general LMAS, we propose a design procedure of the gain matrices in the protocol with respect to a prespecified convergence time and a required consensus accuracy. The design approach is based on a proposed concept of ε-consensus and the application of linear matrix inequalities (LMIs).

We point out that various more complex settings of the consensus problem could be solved by using our linear-transformation-based partial stability approach more spontaneously and effectively, although this paper focuses on the simplest case mentioned above. In fact, we have further solved these problems for discrete-time/continuous linear multi-agent systems with switching communication topologies and without/with time-delay[54, 57] and finished some studies on state consensus of heterogeneous linear multi-agent systems, output consensus problem of high-order linear multi-agent systems and so on.

The rest of the paper is organized as follows. In Section 1 we introduce a description of the consensus problem of the general LMAS and give a consensus definition in terms of an invariant set. Then we propose a linear transformation to set up a bridge between the consensus problem and the partial stability problem. In Section 2, we apply the proposed approach to deal with the consensus problem of general LMAS. In Section 3, we discuss the relation between the gain matrices in the protocol and the convergence time of the agents for the linear systems. Section 4 presents some simulation examples, and Section 5 concludes this paper.

Notations. The following notations are used throughout the paper. Superscript T denotes the transpose of a matrix. $\mathbf{R}^n$ and $\mathbf{R}^{n\times m}$ stand for the real vector space with dimension n and real matrix space of size $n\times m$, respectively. I and 0 mean the identity matrix and zero matrix with compatible dimensions, respectively. Wherever the dimensions of the matrices are not mentioned, they are assumed to be of compatible dimensions.

1 Problem description and its relation to partial asymptotic stability

In this section, we formally describe the consensus problem of the following LMAS

$$\dot{\boldsymbol{x}}_i = A\boldsymbol{x}_i + B\boldsymbol{u}_i, i = 1, \cdots, N \tag{1}$$

where $\boldsymbol{x}_i \in \mathbf{R}^n$ is the state, $\boldsymbol{u}_i \in \mathbf{R}^m$ is the input required to design, and A and B are the matrices of appropriate dimensions.

The control input u_i will be constructed by using state $\boldsymbol{x}_i$ of agent i itself and the relative state $\boldsymbol{x}_j - \boldsymbol{x}_i$ between agent i and the neighboring agent $j \in N_i$, where N_i denotes the index set of neighbors of the agent i. By saying agent j is a neighbor of agent i, we mean that agent j can send its information via communication to agent i. We call set $\{N_i : i = 1, \cdots, N\}$ a communication topology of the LMAS (1).

For a given communication topology $\{N_i : i = 1, \cdots, N\}$, we design a consensus protocol as follows

$$\boldsymbol{u}_i = K\boldsymbol{x}_i + \sum_{j\in N_i} W_{ij}(\boldsymbol{x}_j - \boldsymbol{x}_i) \tag{2}$$

where K and W_{ij} are the gain matrices required to design.

The first term in the protocol of (2) with matrix K is a state feedback of agent i itself and its role consists in changing the final consensus dynamics expressed by the consensus function. The second term with matrices W_{ij} are relative state feedbacks between agent i and its neighbors $j \in N_i$ and their role is to cooperate with the agents for achieving consensus.

Submitting protocol (2) into LMAS (1), we get the equation for the stacked state $\boldsymbol{x} = [\boldsymbol{x}_1^{\mathrm{T}} \;\; \cdots \;\; \boldsymbol{x}_N^{\mathrm{T}}]^{\mathrm{T}}$

$$\dot{\boldsymbol{x}} = (I_N \otimes (A + BK) - (I_N \otimes B)L_W)\boldsymbol{x} \tag{3}$$

where $L_W = [L_{ij}]$ is a weighted block Laplacian with N^2 blocks $L_{ij}, i, j = 1, \cdots, N$ defined as

$$L_{ij} = \begin{cases} \sum\limits_{k=1, k\neq i}^{N} W_{ik}, & j = i \\ -W_{ij}, & j \neq i, j \in N_i \\ 0, & j \neq i, , j \notin N_i \end{cases} \tag{4}$$

We use subspace $\Xi = \{\mathbf{1}_N \otimes \boldsymbol{\xi} : \boldsymbol{\xi} \in \mathbf{R}^n\}$ of the state space $\mathbf{R}^{Nn}$ to give the consensus definition of the multi-agent system (1), where $\otimes$ denotes the Kronecker product of matrices/vectors, and $\mathbf{1}_N \in \mathbf{R}^N$ is the vector with all its components being 1.

Definition 1. Under the given communication topology $\{N_i : i = 1, \cdots, N\}$, the LMAS (1) is called to achieve global consensus via the protocol of (2) if set Ξ is an invariant and globally asymptotically stable set of system (3), i.e., if it satisfies the following properties:

1) Invariance: for any initial state $\boldsymbol{x}(0) \in \Xi$, the trajectory $\boldsymbol{x}(t; \boldsymbol{x}(0))$ starting from $\boldsymbol{x}(0)$ will always remain in set Ξ during the evolution;

2) Global attractability: for any initial state $\boldsymbol{x}(0) \in \mathbf{R}^{Nn}$, trajectory $\boldsymbol{x}(t; \boldsymbol{x}(0))$ converges to set Ξ as $t \to +\infty$;

3) Lyapunov stability: for any $\varepsilon > 0$ there is $\delta > 0$ such that any initial state $\boldsymbol{x}(0) \in \mathbf{R}^{Nn}$ constrained by $d(\boldsymbol{x}(0), \Xi) < \delta$ implies $\sup_{t\geq 0} d(\boldsymbol{x}(t; \boldsymbol{x}(0)), \Xi) < \varepsilon$, where $d(\boldsymbol{x}, \Xi) := \inf_{\boldsymbol{\xi}\in\Xi} \|\boldsymbol{x} - \boldsymbol{\xi}\|$ denotes the distance from point $\boldsymbol{x}$ to set Ξ.

There is a bit of difference between Definition 1 and the consensus definition in the existing literature. We give explanations about it. Property 2) means $\lim_{t\to+\infty} \|\boldsymbol{x}_i - \boldsymbol{x}_j\| = 0$ for any $i, j \in \{1, \cdots, N\}$, which just expresses the consensus meaning in the existing literature. Due to this reason, we call subspace Ξ the consensus set of the LMAS (1). On the other hand, similarly to the concept of Lyapunov stability, for linear dynamic systems, the attractability condition 2) of set Ξ implies its asymptotical stability. However, as well-known, this conclusion is not true for nonlinear dynamic systems, so we need to add the stability condition 3) to attractability to assure the asymptotical stability of set Ξ. Finally, the invariant condition 1) is added since the stability definition for a set is often limited to an invariant set. In fact, the requirements of the stability and invariance of set Ξ are also natural for the consensus issue since the states of the agents should remain either close to or be within the consensus set Ξ if they are close to or have been in it.

For the given communication topology $\{N_i : i = 1, \cdots, N\}$ and protocol (2), we are concerned with two basic issues: 1) to find criteria of consensus convergence, and 2) to calculate the consensus function/value of the agents if the LMAS (1) achieves global consensus.

Now we show how to transform equivalently the consensus problem into a partial asymptotic stability problem of a corresponding system via an appropriate linear transformation of system (3).

The $Nn \times Nn$ linear transformation matrix P is constructed as follows:

$$P := \begin{bmatrix} \tilde{P} \\ \mathbf{1}_N^{\mathrm{T}} \otimes I_n \end{bmatrix}, \tilde{P} = \begin{bmatrix} P_1 \\ \vdots \\ P_{N-1} \end{bmatrix} \tag{5}$$

where I_n is the identity matrix of rank n and $P_i = \begin{bmatrix} p_{i1} & \dots & p_{iN} \end{bmatrix}$ are $n \times Nn$ block matrices with $n \times n$ blocks p_{ij}, $i = 1, \cdots, N-1$, $j = 1, \cdots, N$, satisfying the following two conditions:

1) All the row vectors in each of matrices P_i, $i = 1, \cdots, N-1$, are linearly independent of each other;

2) All the row vectors of P_i are orthogonal to $\mathbf{1}_N^{\mathrm{T}} \otimes I_n$, i.e., $P_i(\mathbf{1}_N \otimes I_n) = 0$, $i = 1, \cdots, N-1$.

The inverse of matrix P is of the following form (see the proof in Appendix).

Lemma 1. The inverse of matrix P defined in (5) is

$$\begin{aligned} &P^{-1} := \begin{bmatrix} \hat{P} & N^{-1}\mathbf{1}_N \otimes I_n \end{bmatrix} \\ &\hat{P} := \begin{bmatrix} \bar{P}_1 & \dots & \bar{P}_{N-1} \end{bmatrix} \\ &\bar{P}_j := \begin{bmatrix} \bar{p}_{1j}^{\mathrm{T}} & \dots & \bar{p}_{Nj}^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}, \ \ j = 1, \cdots, N-1 \end{aligned} \tag{6}$$

and the identity $(\mathbf{1}_N^{\mathrm{T}} \otimes I_n)\hat{P} = 0$ is met, where $\bar{p}_{ij}$, $i = 1, \cdots, N$, $j = 1, \cdots, N-1$ are $n \times n$ blocks indefinitely described.

Using matrix P, we propose the following linear transformation for system (3)

$$\bar{\boldsymbol{x}} = P\boldsymbol{x} \tag{7}$$

Thus, system (3) is transformed into the system

$$\dot{\bar{\boldsymbol{x}}} = PMP^{-1}\bar{\boldsymbol{x}} \tag{8}$$

where

$$M = I_N \otimes (A + BK) - (I_N \otimes B)L_W \tag{9}$$

Letting $\bar{\boldsymbol{x}} = [\boldsymbol{y}^{\mathrm{T}} \quad \boldsymbol{z}^{\mathrm{T}}]^{\mathrm{T}}$, we rewrite system (8) into the form of two equations

$$\begin{aligned} &\dot{\boldsymbol{y}} = \tilde{P}M\hat{P}\boldsymbol{y} + \tilde{P}M(N^{-1}\mathbf{1}_N \otimes I_n)\boldsymbol{z} \\ &\dot{\boldsymbol{z}} = (\mathbf{1}_N^{\mathrm{T}} \otimes I_n)M\hat{P}\boldsymbol{y} + (\mathbf{1}_N^{\mathrm{T}} \otimes I_n)M(N^{-1}\mathbf{1}_N \otimes I_n)\boldsymbol{z} \end{aligned} \tag{10}$$

Definition 2[58]**.** The equilibrium point $\bar{\boldsymbol{x}} = 0$ of system (8) is called to be globally asymptotically stable with respect to the partial variables $\boldsymbol{y}$, or briefly, globally asymptotically $\boldsymbol{y}$-stable, if it satisfies the following two conditions:

1) For any $\varepsilon > 0$, there is a number $\delta > 0$ such that, for the initial state $\bar{\boldsymbol{x}}(0)$, which is arbitrarily given but limited by $\|\boldsymbol{y}(0)\| < \delta$ and $\boldsymbol{z}(0) \in \mathbf{R}^n$, the perturbed trajectory $\bar{\boldsymbol{x}}(t) = [\boldsymbol{y}^{\mathrm{T}}(t) \quad \boldsymbol{z}^{\mathrm{T}}(t)]^{\mathrm{T}}$ satisfies $\sup_{t\geq t_0} \|\boldsymbol{y}(t)\| < \varepsilon$;

2) For any initial state $\bar{\boldsymbol{x}}(0) \in \mathbf{R}^{Nn}$, $\lim_{t\to\infty} \|\boldsymbol{y}(t)\| = 0$.

The proposed linear transformation is motivated by several works in the literature but they are further developed in the paper. For example, motivated by [24] we use the asymptotical stability of an invariant set Ξ to define the consensus. Then the linear transformation P is introduced to make set $P\Xi$ become a basis subspace (e.g. the coordinate axis in the case of one dimension) and thus the asymptotical stability of set Ξ in $\Xi \oplus \Xi_\perp$ becomes that of set $P\Xi$ in the transformed space $P\Xi \oplus P\Xi_\perp$, where $\Xi_\perp$ is the orthogonal subspace to Ξ. The latter is just the partial stability[58] of the equilibrium point 0 in $P\Xi \oplus P\Xi_\perp$ with respect to the coordinate components in $P\Xi_\perp$. As the bases in the subspace of $P\Xi_\perp$, motivated by the error variable method or the contraction theory[59], we often choose the differences of the standard bases in the original space $\Xi_\perp \oplus \Xi$.

Now we are in the position to state a key lemma to express the relation of the consensus problem to the asymptotical y-stability problem (see the proof in Appendix).

Lemma 2. Under the given communication topology $\{N_i : i = 1, \cdots, N\}$, the LMAS (1) achieves global consensus via protocol (2) if and only if the equilibrium point $\bar{\boldsymbol{x}} = 0$ of system (8) is globally asymptotically $\boldsymbol{y}$-stable.

If the global consensus is achieved, the consensus function/value can be calculated by the relation $\boldsymbol{\xi}(t;\boldsymbol{x}(0)) = N^{-1}\boldsymbol{z}(t) = N^{-1}\sum_{i=1}^{N}\boldsymbol{x}_i(t)$.

We will see that the results obtained in the paper is in fact independent of the choice of matrix P. So in practice, matrix P in (5) is often taken in the following typical form

$$P = \left[\begin{array}{c} \tilde{P}_0 \\ \mathbf{1}_N^T \end{array}\right] \otimes I_n, \tilde{P}_0 = \left[\begin{array}{c} \boldsymbol{e}_1 - \boldsymbol{e}_2 \\ \vdots \\ \boldsymbol{e}_{N-1} - \boldsymbol{e}_N \end{array}\right] \tag{11}$$

where $\boldsymbol{e}_i^{\mathrm{T}} \in \mathbf{R}^N$, $i = 1, \cdots, N$ are the standard bases with 1 in the ith row and 0 in all the other rows. In other words, we take the error variables $\bar{\boldsymbol{x}}_i = \boldsymbol{x}_i - \boldsymbol{x}_{i+1}$, $i = 1, \cdots, N-1$ of the original state variables $\boldsymbol{x}_i$, $i = 1, \cdots, N$ as the first $N-1$ new states and the last one is taken as $\bar{\boldsymbol{x}}_N = \sum_{i=1}^{N}\boldsymbol{x}_i$. In this special case, we can exactly calculate the inverse matrix as follows:

$$P^{-1} = \left[\begin{array}{cc} \hat{P}_0 & N^{-1}\mathbf{1}_N \end{array}\right] \otimes I_n$$

$$\hat{P}_0 = \frac{1}{N}\left[\begin{array}{cccc} N-1 & N-2 & \dots & 1 \\ -1 & N-2 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -2 & \dots & 1 \\ -1 & -2 & \dots & -(N-1) \end{array}\right] \tag{12}$$

In the following discussion, we will use this special linear transformation matrix.

Now we state some notations and several lemmas for the sake of future discussion. As shown in the literature, the communication topology $\mathcal{N} = \{N_i : i = 1, \cdots, N\}$ can be presented via a digraph. In fact, letting a vertex set V represent the agents, and a directed edge set $E \subseteq V \times V$ represent the communication topology $\mathcal{N}$ in the way $(j, i) \in E \Leftrightarrow j \in N_i$, one gets a digraph $G = (V, E)$ of the LMAS (1). The communication topology $\mathcal{N}$ is also described by the adjacency matrix $A = [a_{ij}]_{N\times N}$ of the digraph G, whose entries are defined as $a_{ii} = 0$, $a_{ij} = 1$ if $j \in N_i$, otherwise $a_{ij} = 0$. The corresponding standard Laplacian $L = [l_{ij}]$ is defined as $L = D - A$, where $D = \mathrm{diag}\{d_1, \cdots, d_N\}$ is the in-degree matrix with entries $d_i = \sum_{k\in N_i} a_{ik}$. In addition, given the communication topology $\mathcal{N}$ and a weighted matrix $W = [w_{ij}]_{N\times N}$ with positive entries w_{ij}, we also define the weighted in-degree matrix $D_w = \mathrm{diag}\{\sum_{k\in N_1} w_{1k}, \cdots, \sum_{k\in N_N} w_{Nk}\}$ and the weighted adjacency matrix $A_w = A \circ W$ by Hadamard product $\circ$ of matrices. Hence the corresponding weighted Laplacian is $L_w = D_w - A_w$. We use $\mathcal{L}(\mathcal{N})$ to denote the set of all the weighted Laplacians with respect to the communication topology $\mathcal{N}$. One can consider set $\mathcal{L}(\mathcal{N})$ as an equivalent class of the standard Laplacian L with respect to $\mathcal{N}$. The following lemma is well-known (see the proof in Appendix).

Lemma 3. The following statements are equivalent to each other for set $\mathcal{L}(\mathcal{N})$:

1) The digraph $G = (V, E)$ admits a directed spanning tree, i.e., a directed tree covering all the vertices of the digraph;

2) Every $L_w \in \mathcal{L}(\mathcal{N})$ has a simple zero eigenvalue;

3) For every $L_w \in \mathcal{L}(\mathcal{N})$, rank $L_w = N - 1$;

4) For every $L_w \in \mathcal{L}(\mathcal{N})$, matrix $-\tilde{P}_0 L_w \hat{P}_0$ is Hurwitz.

From Lemma 3, we see that any weighted Laplacian L_w satisfies one of the conditions if and only if so does the standard Laplacian L. The stability condition of $-\tilde{P}_0 L \hat{P}_0$ is the easiest to test, especially for the case of high dimensional systems and large number of agents. For example, the Lyapunov equation or Lyapunov inequality as well as Routh criterion can be applied to test the Hurwitz stability of matrix $-\tilde{P}_0 L \hat{P}_0$. Therefore, we use the stability condition of $-\tilde{P}_0 L \hat{P}_0$ to express our results as possible.

2 Criteria of consensus convergence and consensus functions

Now we deal with the consensus problem of the LMAS (1) by using the asymptotical partial stability result for linear systems, and then educe new results on the consensus of the single/double-integrator systems. We also discuss the average consensus problem.

We can verify the matrices in (10) are of the following forms (see the proof in Appendix).

Lemma 4. The following identities are correct:

$$\begin{aligned} &\tilde{P}M(N^{-1}\mathbf{1}_N \otimes I_n) = 0 \\ &(\mathbf{1}_N^{\mathrm{T}} \otimes I_n)M\hat{P} = -(\mathbf{1}_N^{\mathrm{T}} \otimes B)L_W\hat{P} \\ &(\mathbf{1}_N^{\mathrm{T}} \otimes I_n)M(N^{-1}\mathbf{1}_N \otimes I_n) = A + BK \end{aligned}$$

Thus system (10) becomes

$$\begin{aligned} &\dot{\boldsymbol{y}} = \tilde{P}M\hat{P}\boldsymbol{y} \\ &\dot{\boldsymbol{z}} = (A + BK)\boldsymbol{z} - (\mathbf{1}_N^{\mathrm{T}} \otimes B)L_W\hat{P}\boldsymbol{y} \end{aligned} \tag{13}$$

Using Lemma 2 we get the following theorem (see the proof in Appendix).

Theorem 1. Under the given communication topology $\{N_i : i = 1, \cdots, N\}$, the LMAS (1) achieves global consensus via protocol (2) if and only if the matrix $\tilde{P}M\hat{P}$ in (13) is Hurwitz stable. Moreover, the consensus function is expressed by

$$\begin{aligned} \boldsymbol{\xi}(t;\boldsymbol{x}(0)) = N^{-1}\mathrm{e}^{(A+BK)t}\Big(\mathbf{1}_N^{\mathrm{T}} \otimes I_n - \\ \int_0^t \mathrm{e}^{-(A+BK)\tau}(\mathbf{1}_N^{\mathrm{T}} \otimes B)L_W\hat{P}\mathrm{e}^{\tilde{P}M\hat{P}\tau}\tilde{P}\mathrm{d}\tau\Big)\boldsymbol{x}(0) \end{aligned} \tag{14}$$

To test the Hurwitz stability of matrix $\tilde{P}M\hat{P}$, one can apply well-known Routh criterion, Lyapunov equation or Lyapunov inequality.

A special case of protocol (2) is as follows[27,29]

$$\boldsymbol{u}_i = K\boldsymbol{x}_i + W\sum_{j\in N_i} w_{ij}(\boldsymbol{x}_j - \boldsymbol{x}_i) \tag{15}$$

where K and W are the matrices of appropriate dimensions and $w_{ij} > 0$ are scalar weights. In this case, the matrix M in (9) becomes $M = I_N \otimes (A + BK) - L_w \otimes BW$, and thus

$$\tilde{P}M\hat{P} = I_N \otimes (A + BK) - \tilde{P}_0 L_w \hat{P}_0 \otimes BW \tag{16}$$

where L_w is the weighted Laplacian with weights w_{ij}. From Theorem 1, one can get the results obtained in [29].

Corollary 1. Under the given communication topology $\{N_i : i = 1, \cdots, N\}$, the LMAS (1) achieves global consensus via protocol (15) if and only if all the matrices $A + B(K - \lambda_i W)$, $i = 1, \cdots, N-1$ are Hurwitz, where λ_i, $i = 1, \cdots, N-1$ are the eigenvalues of matrix $\tilde{P}_0 L_w \hat{P}_0$. Moreover, the consensus function is expressed by

$$\boldsymbol{\xi}(t;\boldsymbol{x}(0)) = (\boldsymbol{\eta}^{\mathrm{T}} \otimes \mathrm{e}^{(A+BK)t})\boldsymbol{x}(0) \tag{17}$$

where $\boldsymbol{\eta}$ is the left eigenvector of the Laplacian L_w with respect to the zero eigenvalue and satisfies $\boldsymbol{\eta}^{\mathrm{T}}\mathbf{1}_N = 1$.

Now we apply the result of Theorem 1 to the single-integrator multi-agent systems. When $A = 0$ and $B = I_n$, the LMAS (1) becomes

$$\dot{\boldsymbol{x}}_i = \boldsymbol{u}_i,\ \boldsymbol{x}_i \in \mathbf{R}^n,\ i = 1, \cdots, N \tag{18}$$

which is called a single-integrator multi-agent system. We consider the special case of protocol (15)

$$\boldsymbol{u}_i = \sum_{j\in N_i} w_{ij}(\boldsymbol{x}_j - \boldsymbol{x}_i),\ i = 1, \cdots, N \tag{19}$$

By substituting protocol (19) into system (18) and representing them into the vector form, we get the system

$$\dot{\boldsymbol{x}} = -(L_w \otimes I_n)\boldsymbol{x} \tag{20}$$

Thus the corresponding transformed system (13) becomes

$$\begin{aligned}\dot{\boldsymbol{y}} &= -(\tilde{P}_0 L_w \hat{P}_0 \otimes I_n)\boldsymbol{y}\\ \dot{\boldsymbol{z}} &= -(\mathbf{1}_N^{\mathrm{T}} L_w \hat{P}_0 \otimes I_n)\boldsymbol{y}\end{aligned} \tag{21}$$

From Theorem 1, we get the following corollary (see the proof in Appendix).

Corollary 2. Under the given communication topology $\{N_i : i = 1, \cdots, N\}$, the single-integrator multi-agent system (18) achieves global consensus via protocol (19) if and only if $-\tilde{P}_0 L \hat{P}_0$ is Hurwitz stable, where L is the standard Laplacian with respect to the communication topology. Moreover, the consensus value is

$$\boldsymbol{\xi}(\boldsymbol{x}(0)) = N^{-1}\{(\mathbf{1}_N^{\mathrm{T}} - \mathbf{1}_N^{\mathrm{T}} L_w \hat{P}_0(\tilde{P}_0 L_w \hat{P}_0)^{-1}\tilde{P}_0) \otimes I_n\}\boldsymbol{x}(0) \tag{22}$$

As seen in Corollary 2, the criterion of consensus convergence uses a term of Hurwitz stability of matrix $-\tilde{P}_0 L \hat{P}_0$ in stead of that of graph theory, which is helpful to especially verify large scale agent systems. In fact, in addition to the direct calculation of the matrix eigenvalues, we can apply those approaches such as Routh criterion or Lyapunov equation/ineuqality. Another contribution of Corollary 2 is the analytical expression of the consensus value.

The multi-agent system is called achieving average consensus[19], if the consensus value is the average of the initial states of all the agents, i.e., $\boldsymbol{\xi} = N^{-1}\sum_{i=1}^{N}\boldsymbol{x}_i(0)$.

From Corollary 2, it follows that the single-integrator multi-agent system (18) achieves global average consensus via protocol (19) if furthermore one requires $\mathbf{1}_N^{\mathrm{T}} L_w \hat{P}_0(\tilde{P}_0 L_w \hat{P}_0)^{-1}\tilde{P}_0 = 0$, i.e., $\mathbf{1}_N^{\mathrm{T}} L_w \hat{P}_0 = 0$.

Corollary 3. Under the given communication topology $\{N_i : i = 1, \cdots, N\}$, the single-integrator multi-agent system (18) achieves global average consensus via protocol (19) if and only if $-\tilde{P}_0 L \hat{P}_0$ is Hurwitz stable and $\mathbf{1}_N^{\mathrm{T}} L_w \hat{P}_0 = 0$.

If the weighted Laplacian L_w is symmetric, i.e., the graph with respect to the communication topology $\{N_i : i = 1, \cdots, N\}$ is undirected and $w_{ij} = w_{ji}$, one has $\mathbf{1}_N^{\mathrm{T}} L_w = 0$ and thus the second condition in Corollary 3 is naturally satisfied.

Now we apply the result of Theorem 1 to the double-integrator multi-agent systems. When the matrices in (1) are

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \otimes I_n, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes I_n \tag{23}$$

the LMAS (1) is called a double-integrator one

$$\dot{\boldsymbol{x}}_i = A\boldsymbol{x}_i + B\boldsymbol{u}_i, \quad i = 1, \cdots, N \tag{24}$$

where $\boldsymbol{x}_i = [\boldsymbol{r}_i^{\mathrm{T}}\ \boldsymbol{v}_i^{\mathrm{T}}]^{\mathrm{T}} \in \mathbf{R}^{2n}$ is the state, and $\boldsymbol{u}_i \in \mathbf{R}^n$ is the input required to design.

Assume that under the communication topology $\{N_i : i = 1, \cdots, N\}$ the consensus protocol is designed as follows

$$\boldsymbol{u}_i = \sum_{j\in N_i} W_{ij}(\boldsymbol{x}_j - \boldsymbol{x}_i) = \sum_{j\in N_i}\{a_{ij}(\boldsymbol{r}_j - \boldsymbol{r}_i) + b_{ij}(\boldsymbol{v}_j - \boldsymbol{v}_i)\} \tag{25}$$

where $W_{ij} = [a_{ij}\ b_{ij}] \otimes I_n$. Substituting (25) into (24) and writing the closed system into the vector form we get

$$\dot{\boldsymbol{x}} = M\boldsymbol{x} \tag{26}$$

where

$$M = (I_N \otimes \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - L_a \otimes \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} - L_b \otimes \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}) \otimes I_n$$

L_a and L_b are the Laplacians with respect to weights a_{ij} and b_{ij}, respectively.

The corresponding transformed system (13) becomes

$$\dot{\boldsymbol{y}} = (\bar{A} \otimes I_n)\boldsymbol{y}, \quad \dot{\boldsymbol{z}} = A\boldsymbol{z} + (\bar{C} \otimes I_n)\boldsymbol{y} \tag{27}$$

where

$$\begin{aligned}\bar{A} &= I_{N-1} \otimes \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \tilde{P}_0 L_a \hat{P}_0 \otimes \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} - \\ &\quad \tilde{P}_0 L_b \hat{P}_0 \otimes \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \\ \bar{C} &= -\mathbf{1}_N^{\mathrm{T}} L_a \hat{P}_0 \otimes \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} - \mathbf{1}_N^{\mathrm{T}} L_b \hat{P}_0 \otimes \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\end{aligned} \tag{28}$$

From Theorem 1 we get the following corollary (see the proof in Appendix).

Corollary 4. Under the given communication topology $\{N_i : i = 1, \cdots, n\}$, the double-integrator multi-agent system (24) achieves global consensus via protocol (25) if and only if $\bar{A}$ in (28) is Hurwitz stable. Moreover, the consensus function is

$$\begin{aligned}\boldsymbol{\xi}(t; \boldsymbol{x}(0)) = &\frac{1}{N}\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}\Big\{\Big(\mathbf{1}_N^{\mathrm{T}} \otimes I_2 - \bar{C}\bar{A}^{-1}\tilde{P}_0 - \\ &\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\bar{C}\bar{A}^{-2}\tilde{P}_0\Big) \otimes I_n\Big\}\boldsymbol{x}(0)\end{aligned} \tag{29}$$

One can see that in (29) the placement component of the consensus function is a linear function of time t

$$\begin{aligned}&N^{-1}[1\ 0]\Big\{\Big(\mathbf{1}_N^{\mathrm{T}} \otimes I_2 - \bar{C}\bar{A}^{-1}\tilde{P}_0 - \\ &\quad\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\bar{C}\bar{A}^{-2}\tilde{P}_0\Big) \otimes I_n\Big\}\boldsymbol{x}(0) + \\ &\quad N^{-1}[0\ 1]\Big\{\Big(\mathbf{1}_N^{\mathrm{T}} \otimes I_2 - \bar{C}\bar{A}^{-1}\tilde{P}_0\Big) \otimes I_n\Big\}\boldsymbol{x}(0) \cdot t\end{aligned} \tag{30}$$

and the velocity component is a constant vector

$$N^{-1}[0\ 1]\Big\{\Big(\mathbf{1}_N^{\mathrm{T}} \otimes I_2 - \bar{C}\bar{A}^{-1}\tilde{P}_0\Big) \otimes I_n\Big\}\boldsymbol{x}(0) \tag{31}$$

In [20−21, 23], it was assumed that $b_{ij} = \gamma a_{ij}$ in protocol (25), and thus (28) becomes

$$\begin{aligned}\bar{A} &= I_{N-1} \otimes \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \tilde{P}_0 L_a \hat{P}_0 \otimes \begin{bmatrix} 0 & 0 \\ 1 & \gamma \end{bmatrix} \\ \bar{C} &= -\mathbf{1}_N^{\mathrm{T}} L_a \hat{P}_0 \otimes \begin{bmatrix} 0 & 0 \\ 1 & \gamma \end{bmatrix}\end{aligned} \tag{32}$$

The following result was obtained in [23] but it can be deduced directly from Corollary 4 (see the proof in Appendix).

Corollary 5. Under the given communication topology $\{N_i : i = 1, \cdots, N\}$, the double-integrator multi-agent system (24) achieves global consensus via protocol (25) with $b_{ij} = \gamma a_{ij}$ if and only if $-\tilde{P}_0 L \hat{P}_0$ is Hurwitz stable and γ satisfies

$$\gamma > \max_{i=1,\cdots,N-1} \frac{\mathrm{Im}(\mu_i)}{\sqrt{\mathrm{Re}(\mu_i)((\mathrm{Re}(\mu_i))^2 + (\mathrm{Im}(\mu_i))^2)}} \tag{33}$$

where μ_i, $i = 1, \cdots, N-1$, are the eigenvalues of $\tilde{P}_0 L_a \hat{P}_0$, and $\mathrm{Re}(\cdot)$ and $\mathrm{Im}(\cdot)$ denote the real and imaginary parts, respectively.

The condition (33) means that after weights a_{ij} in (25) are chosen arbitrarily, one only needs to choose b_{ij} to be γ times grater than a_{ij} with an enough large γ such that (33) holds.

For the double-integral multi-agent system (24) we can similarly define the concept of velocity average consensus when the velocity component in (31) is the average of the initial velocities of the agents. From Corollary 4, it is easy to derive the following result.

Corollary 6. Under the given communication topology $\{N_i : i = 1, \cdots, N\}$, the double-integrator multi-agent system (24) achieves global velocity average consensus via protocol (25) if and only if $\bar{A}$ in (28) is Hurwitz stable and $[0 \quad 1]\bar{C} = 0$.

If the weighted Laplacians L_a and L_b are symmetric, i.e., the graph with respect to the communication topology $\{N_i : i = 1, \cdots, N\}$ is undirected, $a_{ij} = a_{ji}$ and $b_{ij} = b_{ji}$, then $\bar{C} = 0$, thus the second condition in Corollary 6 is naturally satisfied.

3 Gain matrix design with specified convergence time

In this section, we consider the design problem of the gain matrices in the consensus protocol with respect to a specified consensus convergence time.

Definition 3. Given a small positive number ε and a consensus convergence time T, the LMAS (1) is called to achieve global ε-consensus at time T via protocol (2) under the communication topology $\{N_i : i = 1, \cdots, N\}$ if for any initial state $\boldsymbol{x}(0) = [\boldsymbol{x}_1^{\mathrm{T}}(0) \quad \cdots \ \boldsymbol{x}_N^{\mathrm{T}}(0)]^{\mathrm{T}}$, there exists a function $\boldsymbol{\xi}(t; \boldsymbol{x}(0)) \in \mathbf{R}^n$ related with the initial state such that the state trajectory $\boldsymbol{x}_i(t; \boldsymbol{x}_i(0))$ of agent i starting from the initial state $\boldsymbol{x}_i(0)$ satisfies $\|\boldsymbol{x}_i(t; \boldsymbol{x}_i(0)) - \boldsymbol{\xi}(t; \boldsymbol{x}(0))\| \le \varepsilon$ for any $t \ge \mathrm{T}$, $i = 1, \cdots, N$.

The global ε-consensus is also related with the partial stability. Since for the given initial states of the agents if $\|\boldsymbol{x}(t) - \mathbf{1}_N \otimes \boldsymbol{\xi}(t)\| \le \varepsilon$ then $\|\boldsymbol{x}_i(t) - \boldsymbol{\xi}(t)\| \le \varepsilon$, we can enhance the error requirement by using the former in stead of the latter. On the other hand, the enhanced inequality $\|\boldsymbol{x}(t) - \mathbf{1}_N \otimes \boldsymbol{\xi}(t)\| \le \varepsilon$ is equivalent to $\|P^{-1}[\boldsymbol{y}^{\mathrm{T}} \ 0]^{\mathrm{T}}\| \le \varepsilon$ in terms of the linear transformation $P\boldsymbol{x} = \bar{\boldsymbol{x}} = [\boldsymbol{y}^{\mathrm{T}} \ \boldsymbol{z}^{\mathrm{T}}]^{\mathrm{T}}$ and the relation $\boldsymbol{\xi} = N^{-1}\boldsymbol{z}$. Furthermore, if $\|\boldsymbol{y}\| \le \|P\|\varepsilon$, then $\|P^{-1}[\boldsymbol{y}^{\mathrm{T}} \ 0]^{\mathrm{T}}\| \le \varepsilon$ and thus $\|\boldsymbol{x}_i(t) - \boldsymbol{\xi}(t)\| \le \varepsilon$. That is, we transform the gain design problem of system (1) and (2) with the specified consensus convergence time T into the $\boldsymbol{y}$-stabilization problem of system (8).

We have the following theorem (see the proof in Appendix).

Theorem 2. Given a small positive number ε and a consensus convergence time T, under the given communication topology $\{N_i : i = 1, \cdots, N\}$, the LMAS (1) achieves global ε-consensus at time T via protocol (2) if the matrices K and W_{ij} in protocol (2) are designed such that K and L_W satisfy the following LMI

$$\begin{aligned}&\tilde{P}(I_N \otimes (A + BK) - (I_N \otimes B)L_W)\hat{P} + \\ &\quad \hat{P}^{\mathrm{T}}(I_N \otimes (A + BK) - (I_N \otimes B)L_W)^{\mathrm{T}}\tilde{P}^{\mathrm{T}} \le \\ &\quad -\eta I_{(N-1)n}\end{aligned} \tag{34}$$

where $\eta = \eta(\varepsilon, \ T, \ \boldsymbol{x}(0))$ is an arbitrary positive number satisfying

$$\eta \ge \mathrm{T}^{-1} \ln(\|\boldsymbol{x}(0)\|^2 \varepsilon^{-2}) \tag{35}$$

and $\boldsymbol{x}(0) = [\boldsymbol{x}_1^{\mathrm{T}}(0) \ \cdots \ \boldsymbol{x}_N^{\mathrm{T}}(0)]^{\mathrm{T}}$ are arbitrary initial states.

If we consider protocol (15), the corresponding $\boldsymbol{y}$-stability problem of system (8) becomes the stability problem of $\dot{\boldsymbol{y}} = \tilde{P}M\hat{P}\boldsymbol{y}$, where $\tilde{P}M\hat{P}$ is given in (16). The corresponding LMI (34) becomes

$$\begin{aligned}&I_{N-1} \otimes (A + BK + A^{\mathrm{T}} + K^{\mathrm{T}}B^{\mathrm{T}}) - \tilde{P}_0 L_w \hat{P}_0 \otimes BW - \\ &\quad (\tilde{P}_0 L_w \hat{P}_0)^{\mathrm{T}} \otimes (BW)^{\mathrm{T}} \le -\eta I_{(N-1)n}\end{aligned}$$

Furthermore, for the single-integrator system (18) with protocol (19), where $A = 0$, $B = I_n$, $K = 0$, $L_W = L_w \otimes I_n$, LMI (34) becomes

$$\tilde{P}_0 L_w \hat{P}_0 + (\tilde{P}_0 L_w \hat{P}_0)^{\mathrm{T}} \ge \eta I_{N-1} \tag{36}$$

Corollary 7. Given a small positive number ε and a consensus convergence time T, under the given communication topology $\{N_i : i = 1, \cdots, N\}$, the single-integrator multi-agent system (18) achieves global ε-consensus at time T via protocol (19) if the weighted Laplacian L_w is designed such that all the eigenvalues of $\tilde{P}_0 L_w \hat{P}_0 + (\tilde{P}_0 L_w \hat{P}_0)^{\mathrm{T}}$ are not less than η, where $\eta = \eta(\varepsilon, \ T, \ \boldsymbol{x}(0))$ is an arbitrary positive number satisfying (35).

For the double-integrator system (24) with protocol (25), where A, B are given in (23), we have $K = 0$ and $L_W = L_a \otimes [I_n \quad 0] + L_b[0 \quad I_n]$ and thus LMI (34) becomes

$$(\bar{L}_a + \bar{L}_a^{\mathrm{T}}) + (\bar{L}_b + \bar{L}_b^{\mathrm{T}}) \ge I_{N-1} \otimes \begin{bmatrix} \eta & 1 \\ 1 & \eta \end{bmatrix} \tag{37}$$

where

$$\bar{L}_a = \tilde{P}_0 L_a \hat{P}_0 \otimes \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \bar{L}_b = \tilde{P}_0 L_b \hat{P}_0 \otimes \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \tag{38}$$

Corollary 8. Given a small positive number ε and a consensus convergence time T, under the given communication topology $\{N_i : i = 1, \cdots, N\}$, the double-integrator multi-agent system (24) achieves global ε-consensus at time T via protocol (25) if the weighted Laplacians L_a and L_b are designed such that all the eigenvalues of $\bar{L}_a + \bar{L}_a^{\mathrm{T}} + \bar{L}_b + \bar{L}_b^{\mathrm{T}}$ are not less than $\eta + 1$, where $\eta = \eta(\varepsilon, \ T, \ \boldsymbol{x}(0))$ is an arbitrary positive number satisfying (35).

The positive number η above in fact represents the required convergence velocity of the state with respect to the specified consensus time T and the consensus accuracy ε. One can see that it is relative to the initial state $\boldsymbol{x}(0)$, consensus accuracy ε and consensus time T. The smaller the consensus time T is, the higher the accuracy ε is, and the more dispersive the initial states $\boldsymbol{x}_i(0)$ are, the faster convergence velocity η is required.

As mentioned in Section 1 the gain matrices K and W_{ij} in protocol (2) have different roles and they can be chosen mutually independently. In fact, one can first choose K according to the requirement of the final consensus dynamics expressed in the consensus function and then choose W_{ij} by LMI (34) for a given convergence speed η.

4 Numerical examples

In this section, we consider the consensus verification and consensus design problems of a multi-agent system consisting of 6 agents.

Example 1. Let the matrices in LMAS (1) be

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0.5 \\ 0 & -0.5 & 0 \end{bmatrix}, \ B = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \tag{39}$$

Assume that the communication topology $\{N_i : i = 1, \cdots, 6\}$ is given as

$$N_1 = \{3\}, N_2 = N_3 = \{1\}, N_4 = \{2\}, N_5 = N_6 = \{3\} \tag{40}$$

and the gain matrices in protocol (2) are taken as

$$\begin{aligned} &K = [0\ 0\ 0] \\ &W_{13} = [4\ 6\ 5], W_{21} = [22\ 20\ 23], W_{31} = [18\ 17\ 19] \\ &W_{42} = [18\ 16\ 17], W_{53} = [20\ 22\ 23], W_{63} = [24\ 23\ 25] \end{aligned}$$

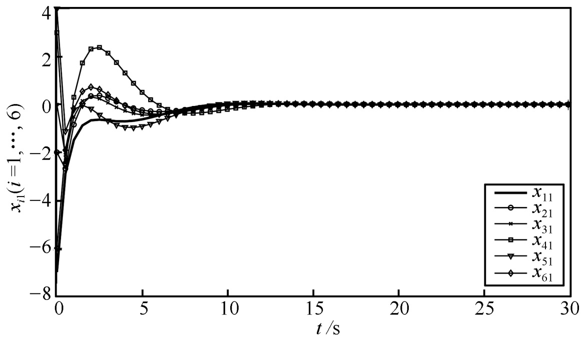
Then the eigenvalues of matrix $\tilde{P}M\hat{P}$ in Theorem 1 are -49.2613, -46.2348, -45.2733, -43.2162, -35.2948, $-0.3526 \pm 0.4868\mathrm{i}$, $-0.3919 \pm 0.4716\mathrm{i}$, $-0.3826 \pm 0.4762\mathrm{i}$, and $-0.3634 \pm 0.4646\mathrm{i}$, $-0.3694 \pm 0.4732\mathrm{i}$. This means that matrix $\tilde{P}M\hat{P}$ is Hurwitz stable. Thus, by Theorem 1, under the communication topology (40), the 6-agent system with matrices (39) achieves global consensus. Figs. 1 (a) $\sim$ (c) show the state trajectories of the three components starting from the initial states $[-7, -3, -2], [-2, 0, 1], [0, -1, 2], [3, 6, 2], [4, -4, 5]$, and $[-6, 0, -5]$, respectively. The corresponding consensus function can be calculated by (14).

Example 2. Now we consider a double integrator 6-agent system (24) with $n = 1$ under the communication topology

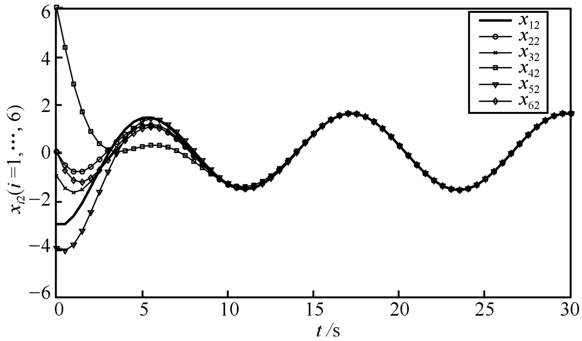
$$\begin{aligned} &N_1 = \{2, 3\}, N_2 = \{1, 4\}, N_3 = \{1, 5, 6\} \\ &N_4 = \{2\}, N_5 = N_6 = \{3\} \end{aligned} \tag{41}$$

Assume that $a_{ij} = 1$, $i = 1, \cdots, 6$, $j \in N_i$ and $b_{ij} = \gamma a_{ij}$ with $\gamma = 2$ in protocol (25). One can verify the conditions in Corollary 6 are satisfied. In fact, the Laplacian L_a is symmetric and the eigenvalues of matrix $\bar{A}$ in (32) are $-7.8948, -5.4495, -2.2813, -0.3249 \pm 0.4683\mathrm{i}, -0.6403, -0.5505, -0.5338, -1.0000$, and -1.0000.

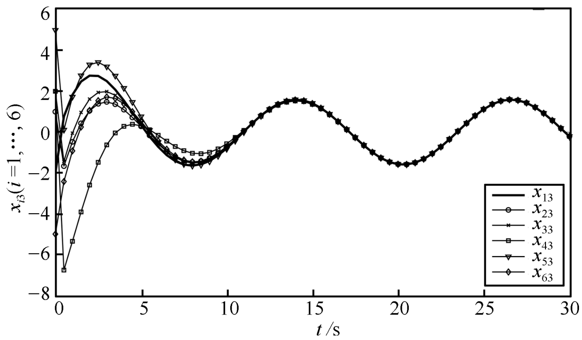
This implies that the double integrator 6 agent system achieves global velocity average consensus. For the initial placements 0, 2, 5, 6, 7, 9, and the initial velocities $0.2, 0.4, 0.6, 0.8, 1.0, 1.2$, Figs. 2 (a) and (b) show the placement and velocity trajectories, respectively. The velocity consensus value is 0.7 and the placement consensus function is $4.833 + 0.7t$.



(a) Trajectories of x_{i1}



(b) Trajectories of x_{i2}



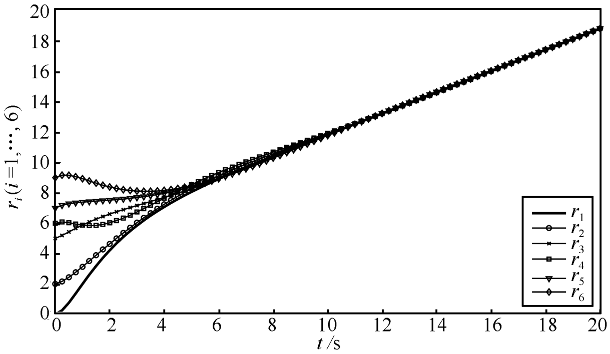
(c) Trajectories of x_{i3}

Fig. 1 State trajectories of the components of the 6-agents

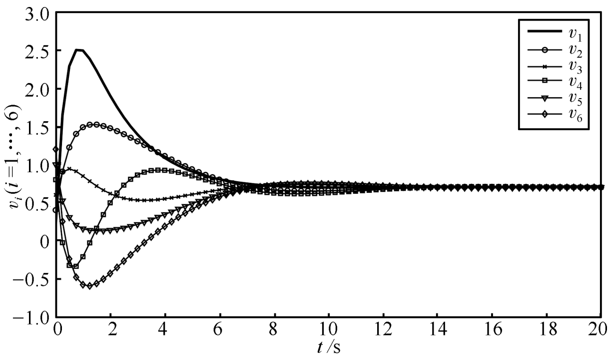
Example 3. Now we consider the design problem of the gain matrices in the protocol for the single/double-integrator 6-agent system with the communication topology in (40), respectively. In both cases, we take the convergence time $T = 10$ and the required error accuracy $\varepsilon = 0.05$.

In the single-integrator 6-agent system, for the initial states $0.2, 0.4, 0.6, 0.8, 1.0, 1.2$, we calculate by (35) the least convergence velocity $\eta = 0.7283$ and use LMI (36) to get weights $w_{13} = 0.4598, w_{21} = 1.2765, w_{31} = 1.0662, w_{42} = 1.1628, w_{53} = 1.4551$, and $w_{63} = 1.4993$. Thus the single-integrator 6-agent system achieves ε-consensus with the consensus value 0.3205. Fig. 3 shows the state trajectories of the 6 agents.

In the double-integrator 6-agent system, for the initial placements $0, 0.2, 0.5, 0.6, 0.7, 0.9$ and the initial velocities $0.2, 0.4, 0.6, 0.8, 1.0, 1.2$, we calculate by (35) the least convergence velocity $\eta = 0.7712$ and use LMI (37) to get weights $w_{13} = 0.2445, w_{21} = 1.0910, w_{31} = 0.8937, w_{42} = 0.8531, w_{53} = 1.1696$, and $w_{63} = 1.1970$. Thus the double-integrator 6-agent system achieves ε-consensus with the velocity consensus value 0.2859 and placement consensus function $0.2859t + 0.1079$. Figs. 4 (a) and (b) show the placement and velocity trajectories of the 6 agents, respectively.



(a) Trajectories of placements



(b) Trajectories of velocities

Fig. 2 State trajectories for the communication topology (41)

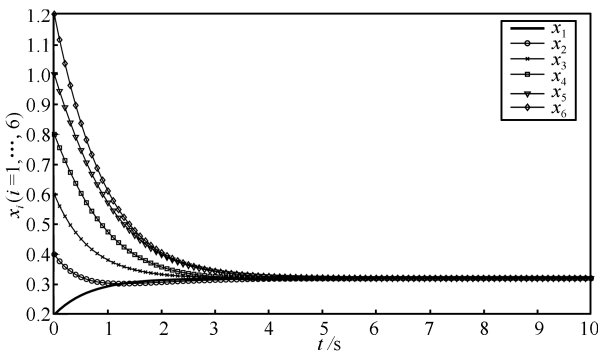
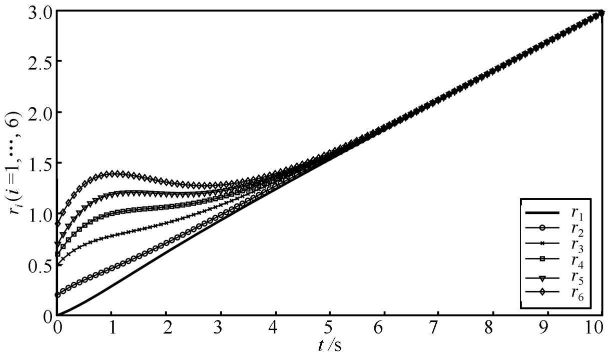


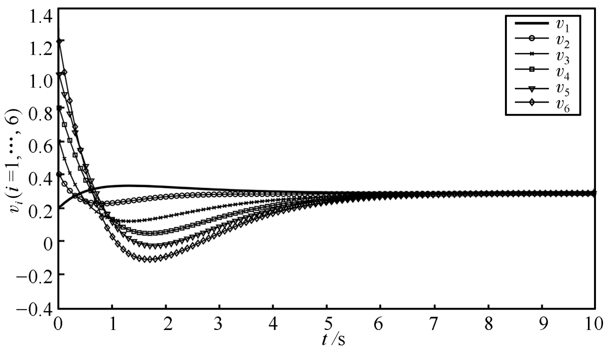
Fig. 3 State trajectories of the single-integrator 6-agent system

5 Conclusion

In the paper, we have set up a bridge between the consensus problem of multi-agent systems and the partial stability problem of corresponding dynamic systems via a linear transformation. By applying the partial stability theory we have given a necessary and sufficient consensus criterion in terms of matrix stability and an analytical expression of the consensus function for general high-order LMASs under a generalized linear consensus protocol. Then applying the results to the consensus and average consensus problems of the single/double-integrator systems, we have improved the existing results in a natural way. The new criteria in terms of matrix stability are easier to test when compared with the existing ones, in particular, for large-scale multi-agent systems since various stability methods can be applied, such as linear matrix equation/inequality, Routh criterion and so on. Finally, we dealt with the design problem of the gain matrices in the protocol. We have proposed an ε-consensus definition for a required consensus convergence time and set up a relation between the gain matrices and the required consensus accuracy ε and the specified consensus convergence time T in terms of linear matrix inequality.



(a) Trajectories of placement



(b) Trajectories of velocity

Fig. 4 State trajectories of the double-integrator 6-agent system

The proposed approach has been shown to be powerful in dealing with the consensus problem, although the study in this paper focuses on the case of the systems with fixed communication topology and in absence of time-delay. In fact, it is not difficult to deal with the various complex settings mentioned in the introduction, which are the topics in the future works.

Appendix

Proof of Lemma 1. Assuming that the inverse of P is $P^{-1} = \left[\bar{P}_1 \cdots \bar{P}_{N-1}\ \bar{P}_N\right]$ with columns $\bar{P}_i$, $i = 1, \cdots, N$, we only need to prove the indentity $\bar{P}_N = N^{-1}\mathbf{1}_N \otimes I_n$.

Since matrix $P^{\rm T}$ is invertible, each column of $\bar{P}_N$ can be linearly represented by the columns of matrix $P^{\rm T}$, so matrix $\bar{P}_N$ can be represented by $q_1 + q_2$, where each column of matrix q_1 can be linearly represented by the columns of matrices $P_i^{\rm T}$, $i = 1, 2, \cdots, N-1$, and q_2 by those of matrix $\mathbf{1}_N \otimes I_n$. Left-multiplying $\bar{P}_N = q_1 + q_2$ by $q_1^{\rm T}$, gets $q_1^{\rm T}\bar{P}_N = q_1^{\rm T}q_1 + q_1^{\rm T}q_2$. Because of $PP^{-1} = I_{Nn}$, one has $P_i\bar{P}_N = 0$, $i = 1, \cdots, N-1$, which implies $q_1^{\rm T}\bar{P}_N = 0$. On the other hand, from the equalities $P_i(\mathbf{1}_N \otimes I_n) = 0$, $i = 1, \cdots, N-1$, one gets $P_i q_2 = 0$, $i = 1, \cdots, N-1$, which means $q_1^{\rm T}q_2 = 0$. Hence, from the equality $q_1^{\rm T}\bar{P}_N = q_1^{\rm T}q_1 + q_1^{\rm T}q_2$ one deduces

$q_1^{\mathrm{T}} q_1 = 0$, i.e., $q_1 = 0$ and thus $\bar{P}_N = q_2$. In other words, one can write $\bar{P}_N$ into $\bar{P}_N = (\mathbf{1}_N \otimes I_n)\alpha$, where α is a matrix of rank $n \times n$. From the identity $P\bar{P} = I_{Nn}$ one has $(\mathbf{1}_N \otimes I_n)^{\mathrm{T}} \bar{P}_N = I_n$, and thus from $\bar{P}_N = (\mathbf{1}_N \otimes I_n)\alpha$ one can get $I_n = N\alpha$, that is, $\alpha = N^{-1} I_n$. Finally, one has the expression $\bar{P}_N = N^{-1}\mathbf{1}_N \otimes I_n$. □

Proof of Lemma 2. First of all, it is easy to verify that under the linear transformation, Ξ is invariant and globally attractive with respect to the system (3) if and only if $P\Xi$ is invariant and globally attractive with respect to system (8). In fact, it is easy to see Ξ is an invariant set of (3) if and only if $P\Xi$ is an invariant set of (8). As for the attractability, if there is $\xi(t)$ such that $\lim_{t\to+\infty} \|\boldsymbol{x}_i(t) - \boldsymbol{\xi}(t)\| = 0$, $i = 1, \cdots, N$, it follows from $\bar{\boldsymbol{x}}_i = P_i(\boldsymbol{x} - \mathbf{1}_N \otimes \boldsymbol{\xi})$, $i = 1, \cdots, N-1$, that $\lim_{t\to+\infty} \|\bar{\boldsymbol{x}}_i(t)\| = 0$, $i = 1, \cdots, N-1$, and therefore the necessary is proved. Conversely, in virtue of Lemma 1 one can verify $\boldsymbol{x}_i = \sum_{j=1}^{N-1} \overline{P}_{ij}\bar{\boldsymbol{x}}_j + N^{-1}\bar{\boldsymbol{x}}_N$, $i = 1, \cdots, N$. So from $\lim_{t\to+\infty} \|\bar{\boldsymbol{x}}_i(t)\| = 0$, $i = 1, \cdots, N-1$, it follows that $\lim_{t\to+\infty} \|\boldsymbol{x}_i(t) - \boldsymbol{\xi}(t)\| = 0$, $i = 1, \cdots, N$, where $\boldsymbol{\xi}(t) = N^{-1}\bar{\boldsymbol{x}}_N(t)$, and thus the sufficiency is verified. Here one can also see $\boldsymbol{\xi}(t) = N^{-1}\bar{\boldsymbol{x}}_N(t) = N^{-1}\sum_{i=1}^{N} \boldsymbol{x}_i(t)$ is just the consensus function.

Next it is direct to prove that Ξ is stable with respect to (3) if and only if $P\Xi$ is stable with respect to (8).

Finally, one can see that the stability of the invariant set $P\Xi$ with respect to (8) is in fact equivalent to the y-stability of equilibrium point 0 of (8) because of $d(\bar{\boldsymbol{x}}, P\Xi) = \|\boldsymbol{y}\|$, where $\bar{\boldsymbol{x}} = [\boldsymbol{y}^{\mathrm{T}}, \bar{\boldsymbol{x}}_N^{\mathrm{T}}]^{\mathrm{T}}$. □

Proof of Lemma 3. For the equivalence of 1) and 2) we refer to Lemma 2.4 in [13]. The equivalence of 2) and 3) is obvious since all the nonzero eigenvalues of L_w are of positive real parts. Finally, one can show that $\tilde{P}_0 L_w \hat{P}_0$ has the same nonzero eigenvalues as those of matrix L_w from the following identity

$$\begin{bmatrix} \tilde{P}_0 \\ \mathbf{1}_N^T \end{bmatrix} L_w \begin{bmatrix} \hat{P}_0 & N^{-1}\mathbf{1}_N \end{bmatrix} = \begin{bmatrix} \tilde{P}_0 L_w \hat{P}_0 & 0 \\ \mathbf{1}_N^T L_w \hat{P}_0 & 0 \end{bmatrix}$$

Hence 4) is equivalent to 2). □

Proof of Lemma 4. Since $L_W(\mathbf{1}_N \otimes I_n) = 0$, $\tilde{P}(\mathbf{1}_N \otimes I_n) = 0$ and $(\mathbf{1}_N^{\mathrm{T}} \otimes I_n)\hat{P} = 0$, we have that

$$\begin{aligned}
&\tilde{P}M(\mathbf{1}_N \otimes I_n) = \\
&\quad \tilde{P}(I_N \otimes (A+BK) - (I_N \otimes B)L_W)(\mathbf{1}_N \otimes I_n) = \\
&\quad \tilde{P}(I_N \otimes (A+BK))(\mathbf{1}_N \otimes I_n) = \\
&\quad \tilde{P}(\mathbf{1}_N \otimes I_n)(A+BK) = 0 \\
&(\mathbf{1}_N^{\mathrm{T}} \otimes I_n)M\hat{P} = \\
&\quad (\mathbf{1}_N^{\mathrm{T}} \otimes I_n)(I_N \otimes (A+BK) - (I_N \otimes B)L_W)\hat{P} = \\
&\quad (\mathbf{1}_N^{\mathrm{T}} \otimes I_n)(I_N \otimes (A+BK))\hat{P} - \\
&\quad (\mathbf{1}_N^{\mathrm{T}} \otimes I_n)(I_N \otimes B)L_W\hat{P} = \\
&\quad (A+BK)(\mathbf{1}_N^{\mathrm{T}} \otimes I_n)\hat{P} - (\mathbf{1}_N^{\mathrm{T}} \otimes I_n)(I_N \otimes B)L_W\hat{P} = \\
&\quad -(\mathbf{1}_N^{\mathrm{T}} \otimes B)L_W\hat{P} \\
&(\mathbf{1}_N^{\mathrm{T}} \otimes I_n)M(\mathbf{1}_N \otimes I_n) = \\
&\quad (\mathbf{1}_N^{\mathrm{T}} \otimes I_n)(I_N \otimes (A+BK) - \\
&\quad (I_N \otimes B)L_W)(\mathbf{1}_N \otimes I_n) = \\
&\quad (\mathbf{1}_N^{\mathrm{T}} \otimes I_n)(I_N \otimes (A+BK))(\mathbf{1}_N \otimes I_n) = \\
&\quad (\mathbf{1}_N^{\mathrm{T}} \otimes I_n)(\mathbf{1}_N \otimes I_n)(A+BK) = N(A+BK)
\end{aligned}$$

Proof of Theorem 1. The sufficient and necessary condition is obtained by applying directly Lemma 2 to the LMAS (1) with protocol (2). We focus the calculation of the consensus function. From the first equation in (14) we get

$$\boldsymbol{y}(t) = \mathrm{e}^{\tilde{P}M\hat{P}t}\boldsymbol{y}(0) = \mathrm{e}^{\tilde{P}M\hat{P}t}\tilde{P}\boldsymbol{x}(0)$$

Substituting $\boldsymbol{y}(t)$ into the second one in (13), we have that

$$\dot{\boldsymbol{z}} = (A+BK)\boldsymbol{z} + \mathbf{1}_N^{\mathrm{T}} M\hat{P}\mathrm{e}^{\tilde{P}M\hat{P}t}\tilde{P}\boldsymbol{x}(0)$$

and obtain expression (14) of the consensus function by solving the equation above and in virtue of Lemma 2. □

Proof of Corollary 1. First of all, one can easily verify that Hurwitz stability of $\tilde{P}M\hat{P}$ is equivalent to Hurwitz stability of all the matrices $A+B(K-\lambda_i W)$, $i = 1, \cdots, N-1$ by transforming $-\tilde{P}_0 L_w \hat{P}_0$ into its Jordan form. We focus on the calculation of the consensus function. In this special case, system (8) becomes $\dot{\boldsymbol{x}} = (I_N \otimes (A+BK) - L_w \otimes BW)\boldsymbol{x}$. So for vector $\boldsymbol{\eta}$ such that $\boldsymbol{\eta}^{\mathrm{T}} L_w = 0$ and $\boldsymbol{\eta}^{\mathrm{T}}\mathbf{1}_N = 1$, we have that

$$(\boldsymbol{\eta}^{\mathrm{T}} \otimes I_n)\dot{\boldsymbol{x}} = (A+BK)(\boldsymbol{\eta}^{\mathrm{T}} \otimes I_n)\boldsymbol{x}$$

When the consensus is achieved we have that $\boldsymbol{x} - \mathbf{1}_N \otimes \boldsymbol{\xi}(t; \boldsymbol{x}(0)) \to 0$ as $t \to \infty$ and thus the consensus function is $\boldsymbol{\xi}(t; \boldsymbol{x}(0)) = (\boldsymbol{\eta}^{\mathrm{T}} \otimes I_n)\boldsymbol{x} = \mathrm{e}^{(A+BK)t}(\boldsymbol{\eta}^{\mathrm{T}} \otimes I_n)\boldsymbol{x}(0)$, which can be written into (17). □

Proof of Corollary 2. By Lemma 2 the linear multi-agent system (18) achieves global consensus via protocol (19) under the given communication topology $\{N_i : i = 1, \cdots, N\}$ if and only if $-(\tilde{P}_0 L_a \hat{P}_0 \otimes I_n)$ is Hurwitz, which, by Lemma 3, is equivalent to Hurwitz stability of $-\tilde{P}_0 L \hat{P}_0$. Now, we calculate the consensus value. Let $\bar{A} = -(\tilde{P}_0 L_a \hat{P}_0 \otimes I_n)$ and $\bar{C} = -(\mathbf{1}_N^{\mathrm{T}} L_a \hat{P}_0 \otimes I_n)$. From the first equation in (21) one gets

$$\boldsymbol{y}(\boldsymbol{t}) = \mathrm{e}^{\bar{A}t}\boldsymbol{y}(0) = \mathrm{e}^{\bar{A}t}\tilde{P}\boldsymbol{x}(0)$$

and from the second one in (21) one has

$$\begin{aligned}
\boldsymbol{z}(t) =& \bar{C}\int_0^t \boldsymbol{y}(\tau)d\tau + \boldsymbol{z}(0) = \\
& \bar{C}\bar{A}^{-1}(\mathrm{e}^{\bar{A}t} - I_{(N-1)n})\tilde{P}\boldsymbol{x}(0) + \boldsymbol{z}(0)
\end{aligned}$$

Thus one obtains the following identity

$$\boldsymbol{z}(t) = \{\bar{C}\bar{A}^{-1}(\mathrm{e}^{\bar{A}t} - I_{(N-1)n)})\tilde{P} + \mathbf{1}_N^{\mathrm{T}} \otimes I_n\}\boldsymbol{x}(0)$$

Since $\bar{A}$ is Hurwitz, the consensus function $\boldsymbol{\xi}(t) = N^{-1}\boldsymbol{z}(t)$ becomes a constant consensus value $N^{-1}\{\mathbf{1}_N^{\mathrm{T}} \otimes I_n - \bar{C}\bar{A}^{-1}\tilde{P}\}\boldsymbol{x}(0)$, which can be written into (22). □

Proof of Corollary 4. The necessary and sufficient condition of global consensus is obtained directly from Theorem 1. Now we focus on the calculation of the consensus function. Let $\tilde{A} = \bar{A} \otimes I_n$ and $\tilde{C} = \bar{C} \otimes I_n$. It follows from the first equation of (27) that $\boldsymbol{y}(t) = \mathrm{e}^{\tilde{A}t}\boldsymbol{y}(0)$. From the second equation of (27) one gets

$$\begin{aligned}
\boldsymbol{z}(t) =& \mathrm{e}^{At}\boldsymbol{z}(0) + \int_0^t \mathrm{e}^{A(t-\tau)}\tilde{C}\boldsymbol{y}(\tau)\mathrm{d}\tau = \\
& \mathrm{e}^{At}\boldsymbol{z}(0) + \mathrm{e}^{At}\int_0^t \mathrm{e}^{-A\tau}\tilde{C}\mathrm{e}^{\tilde{A}\tau}\mathrm{d}\tau\boldsymbol{y}(0)
\end{aligned}$$

We calculate the integral as follows

$$\int_0^t \mathrm{e}^{-A\tau}\tilde{C}\mathrm{e}^{\tilde{A}\tau}\mathrm{d}\tau = \int_0^t \mathrm{e}^{-A\tau}\mathrm{d}(\tilde{C}\mathrm{e}^{\tilde{A}\tau})\tilde{A}^{-1} =$$
$$(\mathrm{e}^{-At}\tilde{C}\mathrm{e}^{\tilde{A}t} - \tilde{C})\tilde{A}^{-1} + A\tilde{C}\int_0^t \mathrm{e}^{\tilde{A}\tau}\mathrm{d}\tau\tilde{A}^{-1} =$$
$$(\mathrm{e}^{-At}\tilde{C}\mathrm{e}^{\tilde{A}t} - \tilde{C})\tilde{A}^{-1} + A\tilde{C}(\mathrm{e}^{\tilde{A}t} - I_{2n(N-1)})\tilde{A}^{-2}$$

Noticing the equality

$$\mathrm{e}^{At} = \left[\begin{array}{cc} 1 & t \\ 0 & 1 \end{array}\right] \otimes I_n$$

we have that $\mathrm{d}(\mathrm{e}^{At})/\mathrm{d}t = A$. Substituting the integral above and the initial conditions $\boldsymbol{y}(0) = (\tilde{P}_0 \otimes I_{2n})\boldsymbol{x}(0)$, $\boldsymbol{z}(0) = (\mathbf{1}_N^{\mathrm{T}} \otimes I_{2n})\boldsymbol{x}(0)$ into $\boldsymbol{z}(t)$, we get

$$\begin{aligned}\boldsymbol{z}(t) =& \mathrm{e}^{At}\Big\{\mathbf{1}_N^{\mathrm{T}} \otimes I_{2n} + \big\{(\mathrm{e}^{-At}\tilde{C}\mathrm{e}^{\tilde{A}t} - \tilde{C})\tilde{A}^{-1} + \\ & A\tilde{C}(\mathrm{e}^{\tilde{A}t} - I_{2(N-1)n})\tilde{A}^{-2}\big\}(\tilde{P}_0 \otimes I_n)\Big\}\boldsymbol{x}(0)\end{aligned}$$

Since $\tilde{A}$ is Hurwitz, i.e., $\lim_{t\to+\infty} \mathrm{e}^{\tilde{A}t} = 0$, the consensus function becomes

$$\begin{aligned}\boldsymbol{\xi}(t;\boldsymbol{x}(0)) =& N^{-1}\mathrm{e}^{At}\Big\{\mathbf{1}_N^{\mathrm{T}} \otimes I_{2n} - \\ & \big\{\tilde{C}\tilde{A}^{-1} + A\tilde{C}\tilde{A}^{-2}\big\}(\tilde{P}_0 \otimes I_n)\Big\}\boldsymbol{x}(0)\end{aligned}$$

Substituting the expressions of A, e^{At}, $\tilde{A}$ and $\tilde{C}$ into $\boldsymbol{\xi}(t;\boldsymbol{x}(0))$ we get the consensus function (29). □

Proof of Corollary 5. Let U be the matrix such that $\tilde{P}_0 L_a \hat{P}_0$ is transformed into its Jordan form J, i.e., $U^{-1}\tilde{P}_0 L_a \hat{P}_0 U = J$. Thus we have

$$(U \otimes I_2)^{-1}\bar{A}(U \otimes I_2) = I_{N-1} \otimes \left[\begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array}\right] - J \otimes \left[\begin{array}{cc} 0 & 0 \\ 1 & \gamma \end{array}\right]$$

From the identity above one can see that if $\bar{A}$ is Hurwitz, then J only has the eigenvalues with positive real parts, or equivalently, $-\tilde{P}_0 L\hat{P}_0$ is Hurwitz in virtue of Lemma 3. Let μ_i, $i = 1, \cdots, N-1$, be all the eigenvalues of matrix $\tilde{P}_0 L_a \hat{P}_0$. Then $\bar{A}$ is Hurwitz if and only if matrices

$$\left[\begin{array}{cc} 0 & 1 \\ -\mu_i & -\gamma\mu_i \end{array}\right], i = 1, \cdots, N-1$$

are Hurwitz. In other words, the roots of the polynomial $\lambda^2 + \gamma\mu_i\lambda + \mu_i$ have negative real parts. Using Lemma 5 we conclude that γ satisfies inequality (33). □

Proof of Theorem 2. Choosing a Lyapunov function $V(\boldsymbol{y}) = \boldsymbol{y}^{\mathrm{T}}\boldsymbol{y}$ For the first subsystem in (13), we have that inequality (34) is equivalent to $\dot{V}(\boldsymbol{y}) \leq -\eta\boldsymbol{y}^{\mathrm{T}}\boldsymbol{y}$. In other words, the first subsystem in (13) is asymptotically stable if (34) is held, which implies the global consensus of the multi-agent system (1) via protocol (2).

Now we determine the positive number η. We define a positive number η_0 as follows

$$\eta_0 = -\min_{\boldsymbol{y}} \frac{\dot{V}(\boldsymbol{y})}{V(\boldsymbol{y})}$$

From this inequality we get $V(\boldsymbol{y}) \leq V(\boldsymbol{y}(0))\mathrm{e}^{-\eta_0 t}$ for any $t \geq 0$, i.e., $\|\boldsymbol{y}\|^2 \leq \|\boldsymbol{y}(0)\|^2\mathrm{e}^{-\eta_0 t}$ for any $t \geq 0$. Thus if $\|\boldsymbol{y}(0)\|^2\mathrm{e}^{-\eta_0 t} \leq \|P\|^2\varepsilon^2$ for any $t \geq T$, or more strictly, if $\|\bar{\boldsymbol{x}}(0)\|^2\mathrm{e}^{-\eta_0 t} \leq \|P\|^2\varepsilon^2$ for any $t \geq T$, we have that $\|\boldsymbol{y}\| \leq \|P\|\varepsilon$ for any $t \geq T$, which implies $\|\boldsymbol{x}_i(t;\boldsymbol{x}_i(0)) - \boldsymbol{\xi}(t;\boldsymbol{x}(0))\| \leq \varepsilon$ for any $t \geq T$, i.e., the multi-agent system (1) achieves global ε -consensus at time T. On the other hand, the condition $\|\bar{\boldsymbol{x}}(0)\|^2\mathrm{e}^{-\eta_0 t} \leq \|P\|^2\varepsilon^2$ for any $t \geq T$ is equivalent to $\eta_0 \geq T^{-1}\ln(\|P\boldsymbol{x}(0)\|^2\|P\|^{-2}\varepsilon^{-2})$. This inequality can be strengthened further as shown in (35) with the help of the inequality $\|P\boldsymbol{x}(0)\| \leq \|P\|\|\boldsymbol{x}(0)\|$. □

References

1 Lynch N A. *Distributed Algorithms*. San Francisco, CA: Morgan Kaufmann, 1997.

2 Wang B. *Coverage Control in Sensor Networks*. London: Springer-Verlag, 2010.

3 Serpedin E, Chaudhari Q M. *Synchronization in Wireless Sensor Networks: Parameter Estimation, Performance Benchmarks and Protocols*. New York: Cambridge University Press, 2009.

4 Cortés J. Finite-time convergent gradient flows with applications to network consensus. *Automatica*, 2006, **42**(11): 1993−2000

5 Olfati-Saber R. Distributed Kalman filtering for sensor networks. In: Proceedings of the 46th IEEE Conference on Decision and Control. New Orleals, LA: IEEE, 2007. 5492−5498

6 Stilwell D J, Bishop B E. Platoons of underwater vehicles. *IEEE Control Systems Magazine*, 2000, **20**(6): 45−52

7 Beard R W, Lawton J, Hadaegh F Y. A feedback architecture for formation control. *IEEE Transactions on Control Systems Technology*, 2001, **9**(1): 777−790

8 Scharf D P, Hadaegh F Y, Ploen S R. A survey of spacecraft formation flying guidance and control (Part II): control. In: Proceedings of the 2004 American Control Conference. Boston, Massachusetts, USA: IEEE, 2004, **4**: 2976−2985

9 Kang W, Yeh H H. Co-ordinated attitude control of multi-satellite systems. *International Journal of Robust and Nonlinear Control*, 2002, **12**(2−3): 185−205

10 Chen Y Q, Wang Z M. Formation control – a review and a new consideration. In: Proceedings of the 2005 IEEE/RSJ International Conference on Intelligent Robots and Systems. Edmonton, Alta: IEEE, 2005. 3181−3186

11 Varaiya P. Smart cars on smart roads: problems of control. *IEEE Transactions on Automatic Control*, 1993, **38**(2): 195−207

12 Mesbahi M, Egerstedt M. *Graph Theoretic Methods in Multiagent Networks*. Princeton: Princeton University Press, 2010

13 Ren W, Beard R W. *Distributed Consensus in Multi-vehicle Cooperative Control*. London: Springer-Verlag, 2008.

14 Ren W, Cao Y C. *Distributed Coordination of Multi-agent Networks*. London: Springer-Verlag, 2011.

15 Olfati-Saber R, Murray R. M. Consensus protocols for networks of dynamic agents. In: Proceedings of the 2003 American Control Conference Boston. Massachusetts, USA: IEEE, 2003. 951−956

16 Lin Z Y, Francis B, Maggiore M. Necessary and sufficient graphical conditions for formation control of unicycles. *IEEE Transactions on Automatic Control*, 2005, **50**(1): 121−127

17 Ren W, Beard R W, McLain T W. Coordination variables and consensus building in multiple vehicle systems. In: Proceedings of the Block Island Workshop on Cooperative Control, Springer-Verlag Series: Lecture Notes in Control and Information Sciences, 2004, **309**: 171−188

18 Wu C W. Agreement and consensus problems in groups of autonomous agents with linear dynamics. In: Proceedings of IEEE International Symposium on Circuits and Systems. Kobe: IEEE, 2005. 292−295

19 Olfati-Saber R, Murray R M. Consensus problems in networks of agents with switching topology and time-delays. *IEEE Transactions on Automatic Control*, 2004, **49**(9): 1520−1533

20 Ren W, Atkins E. Second-order consensus protocols in multiple vehicle systems with local interactions. In: Proceedings of AIAA Guidance, Navigation, and Control Conference and Exhibit. San Francisco, California: AIAA, 2005. 1−13

21 Xie G M, Wang L. Consensus control for a class of networks of dynamic agents. *International Journal of Robust Nonlinear Control*, 2007, **17**(10−11): 941−959

22 Ren W. On consensus algorithms for double-integrator dynamics. *IEEE Transactions on Automatic Control*, 2008, **53**(6): 1503−1509

23 Yu W W, Chen G R, Cao M. Some necessary and sufficient conditions for second-order consensus in multi-agent dynamical systems. *Automatica*, 2010, **46**(6): 1089−1095

24 de Castro G A, Paganini F. Convex synthesis of controllers for consensus. In: Proceedings of the 2004 American Control Conference. Boston, Massachusetts, USA: IEEE, 2004, **6**: 4933−4938

25 Ren W, Moore K L, Chen Y. High-order and model reference consensus algorithms in cooperative control of multivehicle systems. *Journal of Dynamic Systems Measurement and Control*, 2007, **129**(5): 457−462

26 Wang J H, Cheng D Z, Hu X M. Consensus of multi-agent linear dynamic systems. *Asian Journal of Control*, 2008, **10**(2): 144−155

27 Xiao F, Wang L. Consensus problems for high-dimensional multi-agent systems. *IET Control Theory and Applications*, 2007, **1**(3): 830−837

28 Munz U, Papachristodoulou A, Allgower F. Generalized Nyquist consensus condition for high-order linear multiagent systems with communication delays. In: Proceedings of the Joint 48th IEEE Conference on Decision and Control and 28th Chinese Control Conference, Shanghai, China: IEEE, 2009. 4765−4771

29 Xi J X, Cai N, Zhong Y S. Consensus problems for high-order linear time-invariant swarm systems. *Physica A*, 2010, **389**(24): 5619−5627

30 Bliman P A, Trecate G F. Average consensus problems in networks of agents with delayed communications. *Automatica*, 2008, **44**(8): 1985−1995

31 Cepeda-Gomez R, Olgac N. Exhaustive stability analysis in a consensus system with time delay and irregular topologies. *International Journal of Control*, 2011, **84**(4): 746−757

32 Lin P, Jia Y M. Average consensus in networks of multiagents with both switching topology and coupling timedelay. *Physica A: Statistical Mechanics and its Applications*, 2008, **387**(1): 303−313

33 Lin P, Jia Y M. Consensus of a class of second-order multiagent systems with time-delay and jointly-connected topologies. *IEEE Transactions on Automatic Control*, 2010, **55**(3): 778−784

34 Lee D J, Spong M W. Agreement with non-uniform information delays. In: Proceedings of 2006 American Control Conference. Minneapolis, MN: IEEE, 2006. 756−761

35 Munz U, Papachristodoulou A, Allgower F. Nonlinear multiagent system consensus with time-varying delays. In: Proceedings of 17th IFAC World Congress of the International Federation of Automatic Control. Seoul, Korea: IFAC, 2008. 1522−1527

36 Meng Z Y, Ren W, Cao Y C, Zheng Y. Leaderless and leaderfollowing consensus with communication and input delays under a directed network topology. *IEEE Transactions on Systems, Man, and Cybernetics, Part B: Cybernetics*, 2011, **41**(1): 75−88

37 Moreau L. Stability of continuous-time distributed consensus algorithms. In: Proceedings of 43rd IEEE Conference on Decision and Control. Atlantis, Paradise Island, Bahamas: IEEE, 2004, **4**: 3998−4003

38 Sun Y G, Wang L, Xie G M. Average consensus in directed networks of dynamic agents with time-varying delays. In: Proceedings of 45th IEEE Conference on Decision and Control. San Diego, CA: IEEE, 2006. 3393−3398

39 Sun Y G, Wang L, Xie G M. Average consensus in networks of dynamic agents with switching topologies and multiple time-varying delays. *Systems and Control Letters*, 2008, **57**(2): 175−183

40 Sun Y G. Average consensus in networks of dynamic agents with uncertain topologies and time-varying delays. *Journal of the Franklin Institute*, 2012, **349**(3): 1061−1073

41 Tian Y P, Liu C L. Consensus of multi-agent systems with diverse input and communication delays. *IEEE Transactions on Automatic Control*, 2008, **53**(9): 2122−2128

42 Xiao F, Wang L. Asynchronous consensus in continuoustime multiagent systems with switching topology and timevarying delays. *IEEE Transactions on Automatic Control*, 2008, **53**(8): 1804−1816

43 Yang W, Bertozzi A L, Wang X F. Stability of a second order consensus algorithm with time delay. In: Proceedings of the 47th IEEE Conference on Decision and Control. Cancun, Mexico: IEEE, 2008. 2926−2931

44 Pan H, Nian X H, Gui W H. Synchronization in dynamic networks with time-varying delay coupling based on linear feedback controllers. *Acta Automatica Sinica*, 2010, **36**(12): 1766−1772

45 Hong Y G, Gao L X, Cheng D Z, Hu J P. Lyapunov-based approach to multiagent systems with switching jointly connected interconnection. *IEEE Transactions on Automatic Control*, 2007, **52**(5): 943−948

46 Jin Z P, Murray R M. Multi-sop relay protocols for fast consensus seeking. In: Proceedings of 45th IEEE Conference on Decision and Control. San Diego, CA: IEEE, 2006. 1001−1006

47 Kim Y, Mesbahi M. On maximizing the second smallest eigenvalue of a state-dependent graph Laplacian. *IEEE Transactions on Automatic Control*, 2006, **51**(1): 116−120

48 Olshevsky A, Tsitsiklis J N. Convergence rates in distributed consensus and averaging. In: Proceedings of 45th IEEE Conference on Decision and Control. San Diego, CA: IEEE, 2006. 3387−3392

49 Xiao L, Boyd S. Fast linear iterations for distributed averaging. *Systems and Control Letters*, 2004, **53**(1): 65−78

50 Hui Q, Haddad W M, Bhat S P. Finite-time semi-stability and consensus for nonlinear dynamical networks. *IEEE Transactions on Automatic Control*, 2008, **53**(8): 1887−1900

51 Jiang F C, Wang L. Finite-time information consensus for multi-agent systems with fixed and switching topologies. *Physica D: Nonlinear Phenomena*, 2009, **238**(16): 1550−1560

52 Wang L, Xiao F. Finite-time consensus problems for networks of dynamic agents. *IEEE Transactions on Automatic Control*, 2010, **55**(4): 950−955

53 Chen J, Yu M, Dou L H, Gan M G. A fast averaging synchronization algorithm for clock oscillators in nonlinear dynamical network with arbitrary time-delays. *Acta Automatica Sinica*, 2010, **36**(6): 873−880

54 Ge Y R, Chen Y Z, Zhang Y X, He Z H. State consensus analysis and design for high-order discrete-time linear multi-agent systems. *Mathematical Problems in Engineering*, 2013, Article ID 192351, DOI: 10.1155/2013/192351

55 Huang Q Z. Consensus analysis of multi-agent discrete-time systems. *Acta Automatica Sinica*, 2012, **38**(7): 1127−1133

56 Cheng L, Wang Y P, Hou Z G, Tan M, Cao Z Q. Sampled-data based average consensus of second-order integral multi-agent systems: switching topologies and communication noises. *Automatica*, 2013, **49**(5): 1458−1464

57 Chen Y Z, Zhang Y X, He Z H, Ge Y R. Average dwell-time conditions of switching information topologies for consensus of linear multi-agent systems. In: Proceedings of the 32nd Chinese Control Conference. Xi′an, China: IEEE, 2013. 7115−7120

58 Vorotnikov V I. *Partial Stability and Control*. Boston Basel Berlin: Birkhäuser, 1998.

59 Slotine J J E, Wang W. A study of synchronization and group cooperation using partial contraction theory. *Cooperative Control, Lecture Notes in Control and Information Science*, Berlin Heidelberg: Springer, 2004, **309**: 207−228

CHEN Yang-Zhou Professor in the College of Electronic Information and Control Engineering, Beijing University of Technology, China. He received his Ph. D. degree from St. Petersburg University, St. Petersburg, Russia, in 1994. His research interest covers hybrid and network dynamic systems, cooperative control of multiple vehicles/robots, and information processing and control in intelligent transportation systems (ITS). Corresponding author of this paper. E-mail: yzchen@bjut.edu.cn

GE Yan-Rong Ph. D. candidate in the College of Electronic Information and Control Engineering, Beijing University of Technology, China. She is a lecture in the College of Physics Science and Information Engineering, Hebei Normal University, China. Her research interest covers robotic motion planning and cooperation in networked multi-agent systems.
E-mail: gyryanrong@163.com

ZHANG Ya-Xiao Ph. D. candidate in the College of Electronic Information and Control Engineering, Beijing University of Technology, China. Her research interest covers hybrid and network dynamic systems, cooperative control of multiple vehicles/robots, and stability and consensus in networked multi-agent systems.
E-mail: zhangyaxiao@emails.bjut.edu.cn