Robust Delay-dependent H_∞ Consensus Control for Multi-agent Systems with Input Delays

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Abstract This paper investigates the consensus control for multi-agent systems subject to external disturbances, input delays and model uncertainties of networks. By defining an appropriate controlled output, we transform this question into a robust H_∞ control problem. Then, we give two criteria to judge the consensusability of closed-loop multi-agent systems and present a cone-complementary linearization algorithm to get the state feedback controller′s parameters. Finally, numerical examples are given to show the effectiveness of the proposed consensus protocols.

Key words Consensus, multi-agent system, H_∞ control, input delays, undirected graph

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During the last decade, the distributed cooperative control of multi-agent systems has attracted the attention of many researchers from different disciplines[1−9]. This is partly due to its broad applications in different areas, for example, the formation[1], filtering[2], flocking and coordination[3−4], rendezvous problems[5−6] and so forth. In those fields, one of the most important and fundamental issues is consensus control of multi-agent systems which was initially studied for self-driven particles by Vicsek et al.[7]. By using algebraic theory and control theory, Jabdabaie et al.[8] gave an theoretical explanation to Vicsek′s model, and this stirred the excitement of the research on cooperative control in the control community. With the help of abstract system, Tang and Hong[9] used distributed hierarchical control to study the coordination problem of multi-agent systems.

In real systems, time delay phenomena concerning the systems input are often the sources of unconsensusability; this is in general due to the limited communication capacity of systems. Olfati-Saber and Murray[10] studied the consensus problem for first-order multi-agent systems with switching topologies and time-delays. Lin et al.[11] extended the system model proposed in [10] to second-order multi-agent systems. Xiao and Wang[12] studied the consensus problems for discrete-time multi-agent systems with changing communications topologies and bounded time-varying communication delays. Yang and Jia[13] studied the consensus control for linear multi-agent systems with external disturbances and input delays. In real control systems, most of the state information exchanged among agents is analog signal, and there maybe exists signal attenuation during transmission. The operational amplifying circuits will solve this problem, but the amplification factor is affected by the values of resistances and the factor of amplifiers which are not exact values. This kind of uncertainties of communication channel gain is another source of unconsensusability. Zhang and Tian[14] studied the consensus problem for discrete-time linear multi-agent systems with random lossy network and pointed out that the parameters of communication channel would affect the consensusability of multi-agent systems. Mo et al.[15] and Lin et al.[16] studied the consensus problem for multi-agent systems with model uncertainties of networks and input delays.

In this article, we studied the consensus problem for multi-agent systems with linear dynamics, external disturbances, input delays and model uncertainties of networks. Mo et al.[15] gave two theorems to verify that the asymptotic synchronization can be reached with desired H_∞ performance. Lin et al.[16] studied the consensus problem for first-order multi-agent systems with external disturbances, model uncertaint of networks and input delays. By using a different method to deal with the uncertainties of Laplacian, we transform the consensus problem into a robust H_∞ control problem. Two criteria are given to judge the consensus of multi-agent systems in the form of matrix inequalities and an iterative algorithm is given to obtain the state feedback controller. We use the cone-complementary linearization algorithm to cope with the nonlinear matrix inequalities.

1 Problem statement

1.1 Graph theory

In this paper, we use undirected weighted graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$, where $\mathcal{V} = \{v_1, v_2, \cdots, v_n\}$ is the set of nodes, $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the set of undirected edges and $\mathcal{A} = [a_{ij}] \in \mathbf{R}^{n\times n}$ is a symmetric adjacency matrix with weighting factors $a_{ij} \geq 0$ to describe the interaction topology of the multi-agent systems with n agents. An edge of graph $\mathcal{G}$ denoted by pair (v_j, v_i) represents a communication channel between v_j and v_i, and $(v_j, v_i) \in \mathcal{E}$ if and only if $(v_i, v_j) \in \mathcal{E}$. The neighborhood of node v_i is denoted by $N_i = \{v_j \in \mathcal{V} | (v_i, v_j) \in \mathcal{E}\}$. For any $v_i, v_j \in \mathcal{V}$, $a_{ij} = a_{ji} \geq 0$, and $a_{ij} > 0$ if and only if $(v_i, v_j) \in \mathcal{E}$. The Laplacian of a weighted graph $\mathcal{G}$ is defined as $L = D - \mathcal{A}$, where $D = \mathrm{diag}\{d_1, d_2, \cdots, d_n\}$ is a diagonal matrix with $d_i = \sum_{j=1}^n a_{ij}$. A sequence of edges $(v_1, v_2), (v_2, v_3), \cdots, (v_{k-1}, v_k)$ is called a path from node v_1 to node v_k. Graph $\mathcal{G}$ is called a connected graph if for any $v_i, v_j \in \mathcal{V}$, there exists a path from v_i to v_j. Since the Laplacian of an undirected graph is a real symmetric matrix, all its eigenvalues are real numbers.

Lemma 1[17]**.** Consider an undirected graph $\mathcal{G}$ with

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N nodes and M edges. Under the assumption that labels are associated with the edges in a graph whose edges are arbitrarily oriented, the $N \times M$ incidence matrix $D_{\mathcal{G}}$ is a matrix with rows and columns indexed by the nodes and edges of $\mathcal{G}$, such that

$$[D_{\mathcal{G}}]_{ij} = \begin{cases} -1, & \text{if } v_i \text{ is the tail of edge } e_j \\ 1, & \text{if } v_i \text{ is the head of edge } e_j \\ 0, & \text{otherwise} \end{cases}$$

Let $W = \mathrm{diag}\{w_1, w_2, \cdots, w_M\}$, with w_i as the weight of the ith edge of $\mathcal{G}$. Then the Laplacian of $\mathcal{G}$ can be given as

$$L = D_{\mathcal{G}} W D_{\mathcal{G}}^{\mathrm{T}}$$

1.2 Problem formulation

Consider the linear multi-agent system of n interconnected agents with the ith agent modelled by the following dynamic system subject to external disturbances:

$$\dot{\boldsymbol{x}}_i(t) = A\boldsymbol{x}_i(t) + B_1\boldsymbol{u}_i(t) + B_2\boldsymbol{\omega}_i(t) \tag{1}$$

with $\boldsymbol{x}_i \in \mathbf{R}^m$ as the state of ith subsystem, $\boldsymbol{u}_i \in \mathbf{R}^{m_1}$ as the control input, $\boldsymbol{\omega}_i \in \mathbf{R}^{m_2}$ as the external disturbance that belongs to $L_2[0,\infty)$. It is assumed that (A, B_1) is stabilised.

The multi-agent system (1) is said to reach consensus under protocol $\boldsymbol{u}_i(t)$, iff the states of all agents satisfy

$$\lim_{t\to\infty}(\boldsymbol{x}_i(t) - \boldsymbol{x}_j(t)) = 0, \quad \forall i, j \in \{1, 2, \cdots, n\} \tag{2}$$

It may be not easy for us to judge the consensus defined in (2) of multi-agent system (1) under the influence of external disturbances. We define the following controlled output function

$$\boldsymbol{z}_i(t) = \boldsymbol{x}_i(t) - \frac{1}{n}\sum_{j=1}^{n}\boldsymbol{x}_j(t) \tag{3}$$

to measure the disagreement of $\boldsymbol{x}_i(t)$ with the average state value of all agents, $i = 1, 2, \cdots, n$. Note that if $\boldsymbol{z}_i(t) = 0$ for any $i \in \{1, 2, \cdots, n\}$, then $\boldsymbol{x}_i(t) - \boldsymbol{x}_j(t) = 0$ holds for $\forall i \neq j$, that is, the consensus problem of the multi-agent system is solved. Thus, we can use $\boldsymbol{z}_i(t)$ to analyse the consensus behavior of the multi-agent system.

Denote

$$\begin{aligned} \boldsymbol{x}(t) &= [\boldsymbol{x}_1^{\mathrm{T}}(t), \boldsymbol{x}_2^{\mathrm{T}}(t), \cdots, \boldsymbol{x}_n^{T}(t)]^{\mathrm{T}} \in \mathbf{R}^{mn} \\ \boldsymbol{\omega}(t) &= [\boldsymbol{\omega}_1^{\mathrm{T}}(t), \boldsymbol{\omega}_2^{\mathrm{T}}(t), \cdots, \boldsymbol{\omega}_n^{\mathrm{T}}(t)]^{T} \in \mathbf{R}^{m_1 n} \\ \boldsymbol{u}(t) &= [\boldsymbol{u}_1^{\mathrm{T}}(t), \boldsymbol{u}_2^{\mathrm{T}}(t), \cdots, \boldsymbol{u}_n^{\mathrm{T}}(t)]^{\mathrm{T}} \in \mathbf{R}^{m_2 n} \\ \boldsymbol{z}(t) &= [\boldsymbol{z}_1^{\mathrm{T}}(t), \boldsymbol{z}_2^{\mathrm{T}}(t), \cdots, \boldsymbol{z}_n^{\mathrm{T}}(t)]^{\mathrm{T}} \in \mathbf{R}^{mn} \end{aligned}$$

Then the combination of the dynamic equation (1) and the controlled output (3) yields the following system:

$$\begin{aligned} \dot{\boldsymbol{x}}(t) &= (I_n \otimes A)\boldsymbol{x}(t) + (I_n \otimes B_1)\boldsymbol{u}(t) + (I_n \otimes B_2)\boldsymbol{\omega}(t) \\ \boldsymbol{z}(t) &= (L_c \otimes I_m)\boldsymbol{x}(t) \end{aligned} \tag{4}$$

where $L_c(ij) = \frac{n-1}{n}$, $i = j$, otherwise, $L_c(ij) = -\frac{1}{n}$.

Since $\boldsymbol{z}(t) = 0$ means (2) holds, we can use H_∞ norm of the close-loop transfer function $T_{\boldsymbol{z}\boldsymbol{\omega}}(s)$ from the external disturbance $\boldsymbol{\omega}(t)$ to the controlled output $\boldsymbol{z}(t)$, defined as

$$||T_{\boldsymbol{z}\boldsymbol{\omega}}(s)||_\infty = \sup_{v\in\mathbf{R}} \bar{\boldsymbol{\sigma}}(T_{\boldsymbol{z}\boldsymbol{\omega}}(jv)) = \sup_{0\neq\boldsymbol{\omega}(t)\in L_2[0,\infty)} \frac{||\boldsymbol{z}||_2}{||\boldsymbol{\omega}||_2} \tag{5}$$

to measure the attenuating ability of the multi-agent system against external disturbances. Hence, our objective is to design a distributed dynamic output feedback $\boldsymbol{u}_i(t)(i \in \{1, 2, \cdots, n\})$ such that $||T_{\boldsymbol{z}\boldsymbol{\omega}}(s)||_\infty < \gamma$ holds for a given index $\gamma > 0$, or the closed-loop system satisfies the following inequality

$$\int_0^\infty ||\boldsymbol{z}||_2^2 \mathrm{d}t < \gamma^2 \int_0^\infty ||\boldsymbol{\omega}||_2^2 \mathrm{d}t, \quad \forall \boldsymbol{\omega} \in L_2[0,\infty)$$

By doing this, the consensus problem of the multi-agent system subject to external disturbances is transformed into an H_∞ control problem.

1.3 Protocol and model

As information is exchanged between every two agents which are connected by one edge through the communication channel, we must take time delays of message transmission and model uncertainties of networks into account. In this sense, the control protocol of ith agent can only use its neighbour′s lagged states:

$$\boldsymbol{u}_i(t) = \sum_{v_j\in N_i}(a_{ij} + \triangle a_{ij}(t))K[\boldsymbol{x}_j(t-d) - \boldsymbol{x}_i(t-d)] \tag{6}$$

with $d \in [0, \tau]$ as the constant time delay, $K \in \mathbf{R}^{m_1\times m}$ as the state feedback control gain matrix, $\triangle a_{ij}(t)$ as the uncertainty of a_{ij} whose absolute value is much smaller than that of a_{ij} and satisfies $|\triangle\, a_{ij}(t)| \leq \epsilon$. Here, we use $a_{ij}(t)$ to denote the communication channel gain and $\triangle a_{ij}(t)$ the uncertainties of communication channel gain.

Remark 1. Another control protocol of ith agent is stated as follows:

$$\boldsymbol{u}_i(t) = \sum_{v_j\in N_i}(a_{ij} + \triangle a_{ij}(t))K[\boldsymbol{x}_j(t-d) - \boldsymbol{x}_i(t)] \tag{7}$$

That means the ith agent can use its own state information directly without delay, and this kind of protocols have received attention in recent years[3, 6, 18]. Those papers studied consensus problems for first-order and second-order linear systems, where the time delay affects only the information state that is being transmitted. However, the consensus for protocol (7) for a general system with nonuniform time delays remains unknown.

Substituting protocol (6) into system (4) yields the following system:

$$\begin{aligned} \dot{\boldsymbol{x}}(t) =\ & (I_n \otimes A)\boldsymbol{x}(t) + (\widetilde{\mathcal{L}}(t) \otimes B_1K)\boldsymbol{x}(t-d) + \\ & (I_n \otimes B_2)\boldsymbol{\omega}(t) \\ \boldsymbol{z}(t) =\ & (L_c \otimes I_m)\boldsymbol{x}(t) \end{aligned} \tag{8}$$

where $\widetilde{\mathcal{L}} = \mathcal{L} + \triangle\mathcal{L}(t)$ is the Laplacian of graph $\mathcal{G}$.

Let $\bar{\boldsymbol{x}}(t) = \frac{1}{n}\sum_{i=1}^{n}\boldsymbol{x}_i(t)$, and $\boldsymbol{\delta}_i(t) = \boldsymbol{x}_i(t) - \bar{\boldsymbol{x}}(t)$, $i = 1, 2, \cdots, n$. Since $L_c\mathbf{1}_n = 0$ and $\widetilde{\mathcal{L}}(t)\mathbf{1}_n = 0$, we transform system (8) into the following form:

$$\begin{aligned} \dot{\boldsymbol{\delta}}(t) =\ & (L_c \otimes A)\boldsymbol{\delta}(t) + (L_c\widetilde{\mathcal{L}}(t) \otimes B_1K)\boldsymbol{\delta}(t-d) + \\ & (L_c \otimes B_2)\boldsymbol{\omega}(t) \\ \boldsymbol{z}(t) =\ & (L_c \otimes I_m)\boldsymbol{\delta}(t) \end{aligned} \tag{9}$$

where $\boldsymbol{\delta}(t) = [\boldsymbol{\delta}_1^{\mathrm{T}}(t), \boldsymbol{\delta}_2^{\mathrm{T}}(t), \cdots, \boldsymbol{\delta}_n^{\mathrm{T}}(t)]^{\mathrm{T}}$. By introducing the new variable $\boldsymbol{\delta}(t)$, we transform the consensus of multi-agent system (8) into examining the stability of the new system (9).

According to Lemma 2[11], there exists an orthogonal matrix $U = [U_1, U_2] \in \mathbf{R}^{n\times n}(U_2 = \mathbf{1}_n/\sqrt{n})$ such that

$$U^{\mathrm{T}} L_c U = \begin{bmatrix} I_{n-1} & 0_{n-1} \\ * & 0 \end{bmatrix} := \bar{L}_c$$

and

$$U^{\mathrm{T}} \widetilde{\mathcal{L}}(t) U = \begin{bmatrix} L_1(t) & 0_{n-1} \\ * & 0 \end{bmatrix} := \bar{L}_g(t)$$

where $L_1(t)$ is semi-positive definite and has the same nonzero eigenvalues as $\widetilde{\mathcal{L}}(t)$. Pre-multiplying the equation (9) by $U^{\mathrm{T}} \otimes I_m$, we get

$$\begin{aligned} \dot{\bar{\boldsymbol{\delta}}}(t) = \ & (\bar{L}_c \otimes A)\bar{\boldsymbol{\delta}}(t) + (\bar{L}_c \bar{L}_g(t) \otimes B_1 K)\bar{\boldsymbol{\delta}}(t-d) + \\ & (\bar{L}_c \otimes B_2)\bar{\boldsymbol{\omega}}(t) \\ \bar{\boldsymbol{z}}(t) = \ & (\bar{L}_c \otimes I_m)\bar{\boldsymbol{\delta}}(t) \end{aligned} \tag{10}$$

where $\bar{\boldsymbol{\delta}}(t) = (U^{\mathrm{T}} \otimes I_m)\boldsymbol{\delta}(t)$, $\bar{\boldsymbol{\omega}}(t) = (U^{\mathrm{T}} \otimes I_m)\boldsymbol{\omega}(t)$, $\bar{\boldsymbol{z}}(t) = (U^{\mathrm{T}} \otimes I_m)\boldsymbol{z}(t)$. Since the last rows of matrices $\bar{L}_c$ and $\bar{L}_c \bar{L}_g(t)$ are both $\mathbf{0}_n^{\mathrm{T}}$, we get the following reduced-order system from (10):

$$\begin{aligned} \dot{\bar{\boldsymbol{\delta}}}^1(t) = \ & (I_{n-1} \otimes A)\bar{\boldsymbol{\delta}}^1(t) + (L_1(t) \otimes B_1 K)\bar{\boldsymbol{\delta}}^1(t-d) + \\ & (I_{n-1} \otimes B_2)\bar{\boldsymbol{\omega}}^1(t) \\ \bar{\boldsymbol{z}}^1(t) = \ & (I_{n-1} \otimes I_m)\bar{\boldsymbol{\delta}}^1(t) \end{aligned} \tag{11}$$

with $\bar{\boldsymbol{\delta}}^1(t) = (U_1^{\mathrm{T}} \otimes I_m)\boldsymbol{\delta}(t)$, and $\bar{\boldsymbol{\omega}}^1(t) = (U_1^{\mathrm{T}} \otimes I_m)\boldsymbol{\omega}(t)$, and $\bar{\boldsymbol{z}}^1(t) = (U_1^{\mathrm{T}} \otimes I_m)\boldsymbol{z}(t)$. It is easy to prove that system (10) is equivalent to the reduced-order system (11) in terms of H_∞ performance, that is, $||T_{\bar{\boldsymbol{z}}\bar{\boldsymbol{\omega}}}(s)||_\infty = ||T_{\bar{\boldsymbol{z}}^1\bar{\boldsymbol{\omega}}^1}(s)||_\infty$.

Since U is an orthogonal matrix, the transformation between (9) and (10) is an orthogonal one. According to the definition of H_∞ norm defined in (5), we can easily prove that $||T_{\boldsymbol{z}\boldsymbol{\omega}}(s)||_\infty = ||T_{\bar{\boldsymbol{z}}\bar{\boldsymbol{\omega}}}(s)||_\infty = ||T_{\bar{\boldsymbol{z}}^1\bar{\boldsymbol{\omega}}^1}(s)||_\infty$. Moreover, from the orthogonal transformation $\bar{\boldsymbol{\delta}}(t) = (U^{\mathrm{T}} \otimes I_m)\boldsymbol{\delta}(t)$, we know that $\bar{\boldsymbol{\delta}}^1(t) = \mathbf{0}$ results in $\boldsymbol{\delta}(t) = \mathbf{0}$, which guarantees the consensus of multi-agent system (8). Thus, we can study the robust H_∞ consensus behaviour of the multi-agent system (8) by analysising the robust H_∞ performance of the reduced-order system (11).

2 Main results

In this section, we give consensus criteria for the robust H_∞ consensus. Before doing this, we will give a lemma which is useful for the robust H_∞ consensus analysis.

Lemma 2. Let $\tilde{\lambda}_1(t) \le \tilde{\lambda}_2(t) \le \cdots \le \tilde{\lambda}_n(t)$ and $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$ be the eigenvalues of $\widetilde{\mathcal{L}}$ and $\mathcal{L}$ in an ascending order, respectively. Under the assumption that the undirected graph $\mathcal{G}$ is connected, we obtain $0 = \tilde{\lambda}_1(t) < \tilde{\lambda}_2(t) \le \cdots \le \tilde{\lambda}_n(t)$, $0 = \lambda_1 < \lambda_2 \le \cdots \le \lambda_n$ and $\lambda_i - \epsilon\lambda_{\mathcal{G}} \le \tilde{\lambda}_i(t) \le \lambda_i + \epsilon\lambda_{\mathcal{G}}$ with $\lambda_{\mathcal{G}} = ||D_{\mathcal{G}} D_{\mathcal{G}}^{\mathrm{T}}||_2$.

Proof. By the Lemma of 2.4 in [6], under the assumption that graph $\mathcal{G}$ is connected, it is easy to obtain

$$\begin{aligned} & 0 = \tilde{\lambda}_1(t) < \tilde{\lambda}_2(t) \le \cdots \le \tilde{\lambda}_n(t) \\ & 0 = \lambda_1 < \lambda_2 \le \cdots \le \lambda_n \end{aligned}$$

According to Lemma 1, the Laplacian of undirected graph $\mathcal{G}$ can be given as

$$\begin{aligned} \widetilde{\mathcal{L}}(t) = & D_{\mathcal{G}} \widetilde{W}(t) D_{\mathcal{G}}^{\mathrm{T}} = \\ & D_{\mathcal{G}}(W + \triangle W(t)) D_{\mathcal{G}}^{\mathrm{T}} = \\ & \mathcal{L} + \triangle\mathcal{L}(t) \end{aligned}$$

where $\triangle W(t) = \mathrm{diag}\{\triangle w_1(t), \triangle w_2(t), \cdots, \triangle w_M(t)\}$ with $|\triangle w_i(t)| \le \epsilon$, $\forall i \in \{1, 2, \cdots, M\}$. As $|\triangle w_i(t)| \le \epsilon$, we get

$$-\epsilon D_{\mathcal{G}} D_{\mathcal{G}}^{\mathrm{T}} \le D_{\mathcal{G}} \triangle W(t) D_{\mathcal{G}}^{\mathrm{T}} \le \epsilon D_{\mathcal{G}} D_{\mathcal{G}}^{\mathrm{T}}$$

that is

$$|| \triangle \mathcal{L}(t)||_2 \le \epsilon ||D_{\mathcal{G}} D_{\mathcal{G}}^{\mathrm{T}}||_2 = \epsilon\lambda_{\mathcal{G}}$$

We provide a brief constructive proof for the rest proof of the lemma. We assume that for simplicity, $\mathcal{L}$ has n different eigenvalues which satisfy $\epsilon < \frac{\mu}{2\lambda_{\mathcal{G}}}$, where $\mu = \max\{|\lambda_i - \lambda_j| \ |\forall i \ne j\}$. Since $\mathcal{L}$ is a real symmetric matrix, there exists an orthogonal matrix U satisfying $U^{\mathrm{T}}\mathcal{L}U = \Xi$ with $\Xi = \mathrm{diag}\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$. Pre- and post-multiplying the matrix $\widetilde{\mathcal{L}}(t)$ by U^{T} and U, respectively, we get

$$U^{\mathrm{T}} \widetilde{\mathcal{L}}(t) U = U^{\mathrm{T}}(\mathcal{L} + \triangle\mathcal{L}(t))U = \Xi + U^{\mathrm{T}} \triangle \mathcal{L}(t) U$$

By the Corollary of 6.3.4 in [19], the eigenvalues of $\widetilde{\mathcal{L}}(t)$ are contained in the discs:

$$\{z \in \mathbf{C}: \ |z - \lambda_i| \le || \triangle \mathcal{L}(t)||_2 \le \epsilon\lambda_{\mathcal{G}}\}, \quad i = 1, \cdots, n$$

Since $\mu = \max\{|\lambda_i - \lambda_j| \ |\forall i \ne j\}$ and the eigenvalues are continuous functions of the entries of the matrix, there do not exist two discs intersecting each other and each disc containing an eigenvalue of $\widetilde{\mathcal{L}}(t)$. Because the eigenvalues of $\mathcal{L}$ and $\widetilde{\mathcal{L}}(t)$ are real numbers, the discs on the complex plane can be reduced to real intervals:

$$\lambda_i - \epsilon\lambda_{\mathcal{G}} \le \tilde{\lambda}_i(t) \le \lambda_i + \epsilon\lambda_{\mathcal{G}}, \quad i = 1, \cdots, n$$

This completes the proof. □

Remark 2. The symmetric property of Laplacian matrix of a connected undirected graph guarantees that its nonzero eigenvalues are positive real numbers. However, for the connected directed graph, all nonzero eigenvalues of the Laplacian matrix are complex numbers with positve real part. We can use Lemma 2 to deal with the uncertainties of Laplacian matrix and transform the consensus problem into the robust H_∞ control problem of $n-1$ subsystems.

Lemma 3. Assume that the interaction graph $\mathcal{G}$ is connected. System (11) is asymptotically stable and $||T_{\bar{\boldsymbol{z}}^1\bar{\boldsymbol{\omega}}^1}(s)||_\infty < \gamma$, if the following $n-1$ subsystems

$$\begin{aligned} & \dot{\tilde{\boldsymbol{\delta}}}_i(t) = A\tilde{\boldsymbol{\delta}}_i(t) + (\lambda_i + \mu_i(t))B_1 K\tilde{\boldsymbol{\delta}}_i(t-d) + B_2\tilde{\boldsymbol{\omega}}_i(t) \\ & \tilde{\boldsymbol{z}}_i(t) = \tilde{\boldsymbol{\delta}}_i(t), \ i = 2, 3, \cdots, n \end{aligned} \tag{12}$$

are simultaneously asymptotically stable and satisfy the H_∞ performance index γ, where $\tilde{\boldsymbol{\delta}}_i(t)$, $\tilde{\boldsymbol{\omega}}_i(t)$, and $\tilde{\boldsymbol{z}}_i(t)$, will be determined later. λ_i are the positive eigenvalues of $\mathcal{L}$, $\mu_i(t) \in (-\epsilon\lambda_{\mathcal{G}}, \epsilon\lambda_{\mathcal{G}})$.

Proof. By Lemma 2, under the assumption that the undirected graph $\mathcal{G}$ is connected, matrix $L_1(t)$ is positive definite. There exists a unitary matrix $V(t) \in$

$\mathbf{R}^{(n-1)\times(n-1)}$ such that $V^{\mathrm{T}}(t)L_1(t)V(t) = \Lambda(t) := \mathrm{diag}\{\tilde{\lambda}_2(t),\cdots,\tilde{\lambda}_n(t)\}$, where $\tilde{\lambda}_i(t)$ can be rewritten as $\lambda_i+\mu_i(t)$. Denote

$$\begin{aligned}\tilde{\boldsymbol{\delta}}(t) &= (V(t)\otimes I_m)\bar{\boldsymbol{\delta}}^1(t) := [\tilde{\boldsymbol{\delta}}_2^{\mathrm{T}}(t),\cdots,\tilde{\boldsymbol{\delta}}_n^{\mathrm{T}}(t)]^{\mathrm{T}}\\ \tilde{\boldsymbol{\omega}}(t) &= (V(t)\otimes I_m)\bar{\boldsymbol{\omega}}^1(t) := [\tilde{\boldsymbol{\omega}}_2^{\mathrm{T}}(t),\cdots,\tilde{\boldsymbol{\omega}}_n^{\mathrm{T}}(t)]^{\mathrm{T}}\\ \tilde{\boldsymbol{z}}(t) &= (V(t)\otimes I_m)\bar{\boldsymbol{z}}^1(t) := [\tilde{\boldsymbol{z}}_2^{\mathrm{T}}(t),\cdots,\tilde{\boldsymbol{z}}_n^{\mathrm{T}}(t)]^{\mathrm{T}}\end{aligned}$$

with $\tilde{\boldsymbol{\delta}}_i(t)\in\mathbf{R}^m$, $\tilde{\boldsymbol{\omega}}_i(t)\in\mathbf{R}^{m_2}$, $\tilde{\boldsymbol{z}}_i(t)\in\mathbf{R}^m$ $(i=2,\cdots,n)$. The following proof steps are similar to that of Lemma 3.1 in [13], and we omit it. □

For simplicity, we firstly assume that the uncertainties of a_{ij} equal zero, i.e., $\triangle a_{ij}(t)=0$, which indicates $\lambda_i=\tilde{\lambda}_i(t)$, and we use λ_i to replace $\tilde{\lambda}_i(t)$ in Theorem 1. More complicated situation will be given later.

Theorem 1. Assume that the undirected graph $\mathcal{G}$ is connected. Given nonnegative constants τ and γ, the feedback control gain $K=LX^{-1}$ globally asymptotically stabilizes the closed-loop system (12) with an H_∞ disturbance attenuation level of γ, if there exist positive definite matrices X, $\bar{P}_{12}$, $\bar{P}_{22}$, $\bar{Z}_{11}$, $\bar{Z}_{12}$, $\bar{Z}_{22}$, $\bar{Z}_1$, Q, T and matrices L, N_1, N_2, N_3, N_4 with appropriate dimensions such that the following matrix inequalities are satisfied for $i=2,\cdots,n$:

$$\Sigma_i=\begin{bmatrix}\Omega_{i0} & \Xi_{i1}^{\mathrm{T}} & \Xi_{i2}^{\mathrm{T}}\\ * & -T & 0\\ * & * & -XT^{-1}X\end{bmatrix}<0 \tag{13}$$

$$\begin{bmatrix}X & \bar{P}_{12}\\ * & \bar{P}_{22}\end{bmatrix}>0,\quad \begin{bmatrix}\bar{Z}_{11} & \bar{Z}_{12}\\ * & \bar{Z}_{22}-\bar{Z}_1\end{bmatrix}>0 \tag{14}$$

where

$$\Xi_{i1}=[AX\ \lambda_iB_1L\ B_2\ 0\ 0\ 0\ 0\ 0]$$
$$\Xi_{i2}=[0\ 0\ 0\ \bar{P}_{12}\ \tau\bar{Z}_{12}^{\mathrm{T}}\ \tau\bar{Z}_{22}\ 0\ 0]$$

$$\Omega_{i0}=\begin{bmatrix}\hat{\Omega}_{i11} & \hat{\Omega}_{i12} & \hat{\Omega}_{i13} & \hat{\Omega}_{i14} & \hat{\Omega}_{i15} & \hat{\Omega}_{i16} & \tau\bar{N}_1^{\mathrm{T}} & X\\ * & \hat{\Omega}_{i22} & \hat{\Omega}_{i23} & \hat{\Omega}_{i24} & 0 & 0 & \tau\bar{N}_2^{\mathrm{T}} & 0\\ * & * & \hat{\Omega}_{i33} & 0 & 0 & 0 & \tau\bar{N}_3^{\mathrm{T}} & 0\\ * & * & * & \hat{\Omega}_{i44} & 0 & 0 & \tau\bar{N}_4^{\mathrm{T}} & 0\\ * & * & * & * & \hat{\Omega}_{i55} & \hat{\Omega}_{i56} & 0 & 0\\ * & * & * & * & * & \hat{\Omega}_{i66} & 0 & 0\\ * & * & * & * & * & * & \hat{\Omega}_{i77} & 0\\ * & * & * & * & * & * & * & -I\end{bmatrix}$$

with $\hat{\Omega}_{i11}=AX+XA^{\mathrm{T}}+\bar{P}_{12}+\bar{P}_{12}^{\mathrm{T}}-\bar{Z}_{22}+\bar{Q}+\bar{Z}_1+\bar{N}_1+\bar{N}_1^{\mathrm{T}}$, $\hat{\Omega}_{i12}=\lambda_iB_1L-\bar{P}_{12}+\bar{Z}_{22}-\bar{Z}_1-\bar{N}_1^{\mathrm{T}}+\bar{N}_2$, $\hat{\Omega}_{i13}=B_2+\bar{N}_3$, $\hat{\Omega}_{i14}=\bar{P}_{22}-\bar{Z}_{12}^{\mathrm{T}}+\bar{N}_4$, $\hat{\Omega}_{i15}=\tau\bar{Z}_{11}$, $\hat{\Omega}_{i16}=\tau\bar{Z}_{12}$, $\hat{\Omega}_{i22}=-\bar{Z}_{22}-\bar{Q}+\bar{Z}_1-\bar{N}_2-\bar{N}_2^{\mathrm{T}}$, $\hat{\Omega}_{i23}=-\bar{N}_3$, $\hat{\Omega}_{i24}=-\bar{P}_{22}+\bar{Z}_{12}^{\mathrm{T}}-\bar{N}_4$, $\hat{\Omega}_{i33}=-\gamma^2I$, $\hat{\Omega}_{i44}=-\bar{Z}_{11}$, $\hat{\Omega}_{i55}=-\bar{Z}_{11}$, $\hat{\Omega}_{i56}=-\bar{Z}_{12}$, $\hat{\Omega}_{i66}=-\bar{Z}_{22}$, $\hat{\Omega}_{i77}=-\tau\bar{Z}_1$.

Proof. Define the following Lyapunov-Krasovskii function for (12)

$$V_i(\tilde{\boldsymbol{\delta}}_i,t)=\sum_{j=1}^{3}V_{ij}(t)$$

where

$$\begin{aligned}V_{i1}(t)&=\boldsymbol{\eta}_i^{\mathrm{T}}(t)P\boldsymbol{\eta}_i(t)\\ V_{i2}(t)&=d\int_{-d}^{0}\int_{t+\theta}^{t}\boldsymbol{\zeta}_i^{\mathrm{T}}(s)Z\boldsymbol{\zeta}_i(s)\mathrm{d}s\mathrm{d}\theta\\ V_{i3}(t)&=\int_{t-d}^{t}\tilde{\boldsymbol{\delta}}_i^{\mathrm{T}}(s)Q\tilde{\boldsymbol{\delta}}_i(s)\mathrm{d}s\end{aligned}$$

with symmetric positive-definite matrices $Z=\begin{bmatrix}Z_{11} & Z_{12}\\ * & Z_{22}\end{bmatrix}$, P, Q and $\boldsymbol{\eta}_i^{\mathrm{T}}(t)=[\tilde{\boldsymbol{\delta}}_i^{\mathrm{T}}(t),\int_{t-d}^{t}\tilde{\boldsymbol{\delta}}_i^{\mathrm{T}}(s)\mathrm{d}s]$, $\boldsymbol{\zeta}_i^{\mathrm{T}}(s)=[\tilde{\boldsymbol{\delta}}_i^{\mathrm{T}}(s),\dot{\tilde{\boldsymbol{\delta}}}_i^{\mathrm{T}}(s)]$.

The time derivative of $V_i(\tilde{\boldsymbol{\delta}}_i(t),t)$ along the state trajectory of (12) is

$$\dot{V}_i(\tilde{\boldsymbol{\delta}}_i(t),t)=\sum_{j=1}^{3}\dot{V}_i(t) \tag{15}$$

Let us define a new vector variable as

$$\boldsymbol{\chi}_i(t):=\left[\tilde{\boldsymbol{\delta}}_i^{\mathrm{T}}(t),\tilde{\boldsymbol{\delta}}_i^{\mathrm{T}}(t-d),\tilde{\boldsymbol{\omega}}_i^{\mathrm{T}}(t),\int_{t-d}^{t}\tilde{\boldsymbol{\delta}}_i^{\mathrm{T}}(s)\mathrm{d}s\right]^{\mathrm{T}} \tag{16}$$

Then, we define

$$\Gamma_1:=\begin{bmatrix}I & 0 & 0 & 0\\ 0 & 0 & 0 & I\end{bmatrix},\Gamma_{i2}:=\begin{bmatrix}A & \lambda_iB_1K & B_2 & 0\\ I & -I & 0 & 0\end{bmatrix}$$

which allows us to write $\boldsymbol{\eta}_i(t)=\Gamma_1\boldsymbol{\chi}_i(t)$ and $\dot{\boldsymbol{\eta}}_i(t)=\Gamma_{i2}\boldsymbol{\chi}_i(t)$. Hence, we get

$$\dot{V}_{i1}(t)=2\boldsymbol{\eta}_i^{\mathrm{T}}P\dot{\boldsymbol{\eta}}_i(t)=\boldsymbol{\chi}_i^{\mathrm{T}}(t)\Omega_{i1}\boldsymbol{\chi}_i(t) \tag{17}$$

with $\Omega_{i1}:=\Gamma_1^{\mathrm{T}}P\Gamma_{i2}+\Gamma_{i2}^{\mathrm{T}}P\Gamma_1$.

The time derivative of $V_{i2}(t)$ is

$$\dot{V}_{i2}(t)=d^2\boldsymbol{\zeta}_i^{\mathrm{T}}(t)Z\boldsymbol{\zeta}_i(t)-d\int_{t-d}^{t}\boldsymbol{\zeta}_i^{\mathrm{T}}(s)Z\boldsymbol{\zeta}_i(s)\mathrm{d}s$$

According to Jensen integral inequality, we obtain

$$\begin{aligned}&-d\int_{t-d}^{t}\boldsymbol{\zeta}_i^{\mathrm{T}}(s)Z\boldsymbol{\zeta}_i(s)\mathrm{d}s\le\\ &-\left(\int_{t-d}^{t}\boldsymbol{\zeta}_i(s)\mathrm{d}s\right)^{\mathrm{T}}Z\left(\int_{t-d}^{t}\boldsymbol{\zeta}_i(s)\mathrm{d}s\right)\end{aligned} \tag{18}$$

Thus, using (18), we get

$$\begin{aligned}\dot{V}_{i2}(t)\le\ & d^2\boldsymbol{\zeta}_i^{\mathrm{T}}(t)Z\boldsymbol{\zeta}_i(t)-\\ &\int_{t-d}^{t}\boldsymbol{\zeta}_i^{\mathrm{T}}(s)\mathrm{d}s\begin{bmatrix}Z_{11} & Z_{12}\\ * & Z_{22}-Z_1\end{bmatrix}\int_{t-d}^{t}\boldsymbol{\zeta}_i^{\mathrm{T}}(s)\mathrm{d}s-\\ &\left(\int_{t-d}^{t}\dot{\tilde{\boldsymbol{\delta}}}_i(s)\mathrm{d}s\right)^{\mathrm{T}}Z_1\left(\int_{t-d}^{t}\dot{\tilde{\boldsymbol{\delta}}}_i(s)\mathrm{d}s\right)\end{aligned}$$

Define

$$\Gamma_{i3}:=\begin{bmatrix}I & 0 & 0 & 0\\ A & \lambda_iB_1K & B_2 & 0\end{bmatrix},\Gamma_4:=\begin{bmatrix}0 & 0 & 0 & I\\ I & -I & 0 & 0\end{bmatrix}$$

We can denote $\boldsymbol{\zeta}_i(t)=\Gamma_{i3}\boldsymbol{\chi}_i(t)$ and $\int_{t-d}^{t}\boldsymbol{\zeta}_i(s)\mathrm{d}s=\Gamma_4\boldsymbol{\chi}_i(t)$. Note that $d\in[0,\tau]$, we get

$$\dot{V}_{i2}(t)\le\boldsymbol{\chi}_i^{\mathrm{T}}(t)\Omega_{i2}\boldsymbol{\chi}_i(t) \tag{19}$$

with $\Omega_{i2} := \tau^2\Gamma_{i3}^{\mathrm{T}} Z\Gamma_{i3} - \Gamma_4^{\mathrm{T}} \begin{bmatrix} Z_{11} & Z_{12} \\ * & Z_{22}-Z_1 \end{bmatrix} \Gamma_4$.

Define $\Gamma_5 := [I\ 0\ 0\ 0]$, $\Gamma_6 := [0\ I\ 0\ 0]$ and

$$\Omega_{i3} := \Gamma_5^{\mathrm{T}} Q\Gamma_5 - \Gamma_6^{\mathrm{T}} Q\Gamma_6$$

We can rewrite $\tilde{\boldsymbol{\delta}}_i(t) = \Gamma_5\boldsymbol{\chi}_i(t)$, $\tilde{\boldsymbol{\delta}}_i(t-d) = \Gamma_6\boldsymbol{\chi}_i(t)$ and get

$$\dot{V}_{i3}(t) = \boldsymbol{\chi}_i^{\mathrm{T}}(t)\Omega_{i3}\boldsymbol{\chi}_i(t) \tag{20}$$

Using equation $\int_{t-d}^{t}\dot{\tilde{\boldsymbol{\delta}}}_i(s)\mathrm{d}s = \tilde{\boldsymbol{\delta}}_i(t) - \tilde{\boldsymbol{\delta}}_i(t-d)$, we can construct the following null equation:

$$2\boldsymbol{\chi}_i^{\mathrm{T}}(t)N^{\mathrm{T}}\left[\Gamma_7\boldsymbol{\chi}_i(t) - \int_{t-d}^{t}\dot{\tilde{\boldsymbol{\delta}}}_i(s)\mathrm{d}s\right] = 0 \tag{21}$$

with $N := [N_1\ N_2\ N_3\ N_4]$ and $\Gamma_7 := [I\ -I\ 0\ 0]$. Completing (21) to squares results in

$$0 \le \boldsymbol{\chi}_i^{\mathrm{T}}(t)\Omega_{i4}\boldsymbol{\chi}_i(t) + \left(\int_{t-d}^{t}\dot{\tilde{\boldsymbol{\delta}}}_i(s)\mathrm{d}s\right)^{\mathrm{T}} Z_1 \left(\int_{t-d}^{t}\dot{\tilde{\boldsymbol{\delta}}}_i(s)\mathrm{d}s\right) \tag{22}$$

where $\Omega_{i4} := N^{\mathrm{T}}\Gamma_7 + \Gamma_7^{\mathrm{T}}N + \tau N^{\mathrm{T}}Z_1^{-1}N$. Adding (22) to $\dot{V}_i(\tilde{\boldsymbol{\delta}}_i(t),t)$ and substituting $\dot{V}_{ij}(t)$, $j=1,2,3$ computed in (17), (19) and (20) yields

$$\dot{V}_i(\tilde{\boldsymbol{\delta}}_i(t),t) + \tilde{\boldsymbol{z}}_i^{\mathrm{T}}(t)\tilde{\boldsymbol{z}}_i(t) - \gamma^2\tilde{\boldsymbol{\omega}}_i^{\mathrm{T}}(t)\tilde{\boldsymbol{\omega}}_i(t) \le \boldsymbol{\chi}_i^{\mathrm{T}}(t)\Omega_i\boldsymbol{\chi}_i(t) \tag{23}$$

where $\Omega_i = (\sum_{j=1}^{4}\Omega_{ij}) + \Gamma_5^{\mathrm{T}}\Gamma_5 - \gamma^2\Gamma_8^{\mathrm{T}}\Gamma_8$ with $\Gamma_8 := [0\ 0\ I\ 0]$ and is explicitly calculated as

$$\Omega_i = \begin{bmatrix} \Omega_{i11} & \Omega_{i12} & \Omega_{i13} & \Omega_{i14} \\ * & \Omega_{i22} & \Omega_{i23} & \Omega_{i24} \\ * & * & \Omega_{i33} & \Omega_{i34} \\ * & * & * & \Omega_{i44} \end{bmatrix} < 0 \tag{24}$$

where $\Omega_{i11} = P_{11}A + A^{\mathrm{T}}P_{11} + P_{12} + P_{12}^{\mathrm{T}} + \tau^2(Z_{11} + Z_{12}A + A^{\mathrm{T}}Z_{12}^{\mathrm{T}} + A^{\mathrm{T}}Z_{22}) + Z_1 + N_1 + N_1^{\mathrm{T}} + \tau N_1^{\mathrm{T}}Z_1^{-1}N_1 - Z_{22} + Q + I$, $\Omega_{i12} = \lambda_i P_{11}B_1K - P_{12} + \tau^2(\lambda_i Z_{12}B_1K + \lambda_i A^{\mathrm{T}}Z_{22}B_1K) + Z_{22} - Z_1 - N_1^{\mathrm{T}} + N_2 + \tau N_1^{\mathrm{T}}Z_1^{-1}N_2$, $\Omega_{i13} = P_{11}B_2 + \tau^2(Z_{12}B_2 + A^{\mathrm{T}}Z_{22}B_2) + N_3 + \tau N_1^{\mathrm{T}}Z_1^{-1}N_3$, $\Omega_{i14} = A^{\mathrm{T}}P_{12} + P_{22} - Z_{12}^{\mathrm{T}} + N_4^{\mathrm{T}} + \tau N_1^{\mathrm{T}}Z_1^{-1}N_4$, $\Omega_{i22} = \tau^2\lambda_i^2K^{\mathrm{T}}B_1^{\mathrm{T}}Z_{22}B_1K - Z_{22} - Q + Z_1 - N_2 - N_2^{\mathrm{T}} + \tau N_2^{\mathrm{T}}Z_1^{-1}N_2$, $\Omega_{i23} = \tau^2\lambda_iK^{\mathrm{T}}B_1^{\mathrm{T}}Z_{22}B_2 - N_3 + \tau N_2^{\mathrm{T}}Z_1^{-1}N_3$, $\Omega_{i24} = \lambda_iK^{\mathrm{T}}B_1^{\mathrm{T}}P_{12} - P_{22} + Z_{12}^{\mathrm{T}} - N_4 + \tau N_2^{\mathrm{T}}Z_1^{-1}N_4$, $\Omega_{i33} = \tau^2B_2^{\mathrm{T}}Z_{22}B_2 - \gamma^2I + \tau N_3^{\mathrm{T}}Z_1^{-1}N_3$, $\Omega_{i34} = B_2^{\mathrm{T}}P_{12} + \tau N_3^{\mathrm{T}}Z_1^{-1}N_4$, $\Omega_{i44} = \tau N_4^{\mathrm{T}}Z_1^{-1}N_4 - Z_{11}$.

If $\Omega_i < 0$, then $\dot{V}_i(\tilde{\boldsymbol{\delta}}_i(t),t) + \tilde{\boldsymbol{z}}_i^{\mathrm{T}}(t)\tilde{\boldsymbol{z}}_i(t) - \gamma^2\tilde{\boldsymbol{\omega}}_i^{\mathrm{T}}(t)\tilde{\boldsymbol{\omega}}_i(t) \le \boldsymbol{\chi}_i^{\mathrm{T}}(t)\Omega_i\boldsymbol{\chi}_i(t) < 0$. When $\tilde{\boldsymbol{\omega}}_i(t) \equiv 0$, $\forall t \ge 0$, $\dot{V}_i(\tilde{\boldsymbol{\delta}}_i(t),t)$ still holds and guarantees that system (12) with no disturbance is globally asymptotically stable. Hence, the initial state is supposed to be zero-valued, under which we consider the cost function

$$\begin{aligned} J_{iT} = &\int_0^T [\tilde{\boldsymbol{z}}_i^{\mathrm{T}}(t)\tilde{\boldsymbol{z}}_i(t) - \gamma^2\tilde{\boldsymbol{\omega}}_i^{\mathrm{T}}(t)\tilde{\boldsymbol{\omega}}_i(t) + \dot{V}_i(\tilde{\boldsymbol{\delta}}_i(t),t)]\mathrm{d}t - \\ &V_i(\tilde{\boldsymbol{\delta}}_i(T),T) + V_i(0,0) \le \\ &\int_0^T \boldsymbol{\chi}_i^{\mathrm{T}}(t)\Omega_i\boldsymbol{\chi}_i(t)\mathrm{d}t < 0 \end{aligned}$$

that is, $\int_0^T ||\tilde{\boldsymbol{z}}_i(t)||^2\mathrm{d}t < \gamma^2\int_0^T ||\tilde{\boldsymbol{\omega}}_i(t)||^2\mathrm{d}t$. Let $T \to \infty$, we get $||\tilde{\boldsymbol{z}}_i||_2 < \gamma||\tilde{\boldsymbol{\omega}}_i||_2$, $\forall\tilde{\boldsymbol{\omega}}_i(t) \in L_2[0,\infty)$. Applying Schur complement formula to $\Omega_i < 0$ in (24), we get $\bar{\Omega}_i < 0$

$$\begin{bmatrix} \bar{\Omega}_{i11} & \bar{\Omega}_{i12} & \bar{\Omega}_{i13} & \bar{\Omega}_{i14} & \bar{\Omega}_{i15} & \bar{\Omega}_{i16} & \tau N_1^{\mathrm{T}} & I \\ * & \bar{\Omega}_{i22} & \bar{\Omega}_{i23} & \bar{\Omega}_{i24} & \bar{\Omega}_{i25} & \bar{\Omega}_{i26} & \tau N_2^{\mathrm{T}} & 0 \\ * & * & \bar{\Omega}_{i33} & \bar{\Omega}_{i34} & \bar{\Omega}_{i35} & \bar{\Omega}_{i36} & \tau N_3^{\mathrm{T}} & 0 \\ * & * & * & \bar{\Omega}_{i44} & 0 & 0 & \tau N_4^{\mathrm{T}} & 0 \\ * & * & * & * & \bar{\Omega}_{i55} & \bar{\Omega}_{i56} & 0 & 0 \\ * & * & * & * & * & \bar{\Omega}_{i66} & 0 & 0 \\ * & * & * & * & * & * & \bar{\Omega}_{i77} & 0 \\ * & * & * & * & * & * & * & -I \end{bmatrix} \tag{25}$$

with $\bar{\Omega}_{i11} = P_{11}A + A^{\mathrm{T}}P_{11} + P_{12} + P_{12}^{\mathrm{T}} - Z_{22} + Q + Z_1 + N_1 + N_1^{\mathrm{T}}$, $\bar{\Omega}_{i12} = \lambda_iP_{11}B_1K - P_{12} + Z_{22} - Z_1 - N_1^{\mathrm{T}} + N_2$, $\bar{\Omega}_{i13} = P_{11}B_2 + N_3$, $\bar{\Omega}_{i14} = A^{\mathrm{T}}P_{12} + P_{22} - Z_{12}^{\mathrm{T}} + N_4^{\mathrm{T}}$, $\bar{\Omega}_{i15} = \tau Z_{11} + \tau A^{\mathrm{T}}Z_{12}^{\mathrm{T}}$, $\bar{\Omega}_{i16} = \tau Z_{12} + \tau A^{\mathrm{T}}Z_{22}$, $\bar{\Omega}_{i22} = -Z_{22} - Q + Z_1 - N_2 - N_2^{\mathrm{T}}$, $\bar{\Omega}_{i23} = -N_3$, $\bar{\Omega}_{i24} = \Omega_{i24} - \tau N_2^{\mathrm{T}}Z_1^{-1}N_4$, $\bar{\Omega}_{i25} = \tau\lambda_iK^{\mathrm{T}}B_1^{\mathrm{T}}Z_{12}^{\mathrm{T}}$, $\bar{\Omega}_{i26} = \tau\lambda_iK^{\mathrm{T}}B_1^{\mathrm{T}}Z_{22}$, $\bar{\Omega}_{i33} = -\gamma^2I$, $\bar{\Omega}_{i34} = B_2^{\mathrm{T}}P_{12}$, $\bar{\Omega}_{i35} = \tau B_2^{\mathrm{T}}Z_{12}^{\mathrm{T}}$, $\bar{\Omega}_{i36} = \tau B_2Z_{22}$, $\bar{\Omega}_{i44} = -Z_{11}$, $\bar{\Omega}_{i55} = -Z_{11}$, $\bar{\Omega}_{i56} = -Z_{12}$, $\bar{\Omega}_{i66} = -Z_{22}$, $\bar{\Omega}_{i77} = -\tau Z_1$.

Let $U = \mathrm{diag}\{X, X, I, X, X, X, X, I\}$ where $X := P_{11}^{-1}$. Pre and post-multiplying matrix $\bar{\Omega}_i$ by U, i.e $\hat{\Omega}_i = U\bar{\Omega}_iU < 0$, and employing the variable changes $\overline{(\cdot)} := X(\cdot)X$ except for variable N_3 for which we define it as $\bar{N}_3 := XN_3$, we get

$$\begin{bmatrix} \hat{\Omega}_{i11} & \hat{\Omega}_{i12} & \hat{\Omega}_{i13} & \tilde{\Omega}_{i14} & \tilde{\Omega}_{i15} & \tilde{\Omega}_{i16} & \tau\bar{N}_1^{\mathrm{T}} & X \\ * & \hat{\Omega}_{i22} & \hat{\Omega}_{i23} & \tilde{\Omega}_{i24} & \tilde{\Omega}_{i25} & \tilde{\Omega}_{i26} & \tau\bar{N}_2^{\mathrm{T}} & 0 \\ * & * & \hat{\Omega}_{i33} & \tilde{\Omega}_{i34} & \tilde{\Omega}_{i35} & \tilde{\Omega}_{i36} & \tau\bar{N}_3^{\mathrm{T}} & 0 \\ * & * & * & \hat{\Omega}_{i44} & 0 & 0 & \tau\bar{N}_4^{\mathrm{T}} & 0 \\ * & * & * & * & \hat{\Omega}_{i55} & \hat{\Omega}_{i56} & 0 & 0 \\ * & * & * & * & * & \hat{\Omega}_{i66} & 0 & 0 \\ * & * & * & * & * & * & \hat{\Omega}_{i77} & 0 \\ * & * & * & * & * & * & * & -I \end{bmatrix} \tag{26}$$

where $\tilde{\Omega}_{i14} = \hat{\Omega}_{i14} + XA^{\mathrm{T}}X^{-1}\bar{P}_{12}$, $\tilde{\Omega}_{i15} = \hat{\Omega}_{i15} + \tau XA^{\mathrm{T}}X^{-1}\bar{Z}_{12}^{\mathrm{T}}$, $\tilde{\Omega}_{i16} = \hat{\Omega}_{i16} + \tau XA^{\mathrm{T}}X^{-1}\bar{Z}_{22}$, $\tilde{\Omega}_{i24} = \hat{\Omega}_{i24} + \lambda_iL^{\mathrm{T}}B_1^{\mathrm{T}}X^{-1}\bar{P}_{12}$, $\tilde{\Omega}_{i25} = \tau\lambda_iL^{\mathrm{T}}B_1^{\mathrm{T}}X^{-1}\bar{Z}_{12}^{\mathrm{T}}$, $\tilde{\Omega}_{i26} = \tau\lambda_iL^{\mathrm{T}}B_1^{\mathrm{T}}X^{-1}\bar{Z}_{22}$, $\tilde{\Omega}_{i34} = B_2^{\mathrm{T}}X^{-1}\bar{P}_{12}$, $\tilde{\Omega}_{i35} = \tau B_2^{\mathrm{T}}X^{-1}\bar{Z}_{12}^{\mathrm{T}}$, $\tilde{\Omega}_{i36} = \tau B_2X^{-1}\bar{Z}_{22}$, with $L := KX$.

Note that $\hat{\Omega}_i$ can be decomposed into

$$\hat{\Omega}_i = \Omega_{i0} + \Xi_{i1}^{\mathrm{T}}X^{-1}\Xi_{i2} + \Xi_{i2}^{\mathrm{T}}X^{-1}\Xi_{i1}$$

and for any positive definite matrix T, we have the following inequality

$$\begin{aligned} &\Xi_{i1}^{\mathrm{T}}X^{-1}\Xi_{i2} + \Xi_{i2}^{\mathrm{T}}X^{-1}\Xi_{i1} \le \\ &\Xi_{i1}^{\mathrm{T}}T^{-1}\Xi_{i1} + \Xi_{i2}^{\mathrm{T}}(XT^{-1}X)^{-1}\Xi_{i2} \end{aligned}$$

Substituting the above inequality into $\hat{\Omega}_i$ and applying Schur complement formula leads to (13). □

Due to the existence of nonlinear entry $-XT^{-1}X$, the matrix inequality conditions (13) are not LMIs. In order to obtain the feedback gain K, we employ the so-called cone-complementary linearization algorithm[20−21]. First, we define a new positive definite matrix W such that $W \le$

$XT^{-1}X$ and replace inequality conditions (13) with

$$\begin{bmatrix} \Omega_{i0} & \Xi_{i1}^{\mathrm{T}} & \Xi_{i2}^{\mathrm{T}} \\ * & -T & 0 \\ * & * & -W \end{bmatrix} < 0 \tag{27}$$

and

$$\begin{bmatrix} \bar{W} & \bar{X} \\ * & \bar{T} \end{bmatrix} \geq 0, \begin{bmatrix} \bar{W} & I \\ * & W \end{bmatrix} \geq 0, \begin{bmatrix} \bar{X} & I \\ * & X \end{bmatrix} \geq 0, \begin{bmatrix} \bar{T} & I \\ * & T \end{bmatrix} \geq 0 \tag{28}$$

where $\bar{W}$, $\bar{X}$ and $\bar{T}$ are positive definite matrices with appropriate dimensions. Then the feedback gain matrix can be obtained by the following linearized algorithm[20−21]:

1) Choose a large initial value for γ and small one for τ such that there exists a solution set $\{\bar{X}_0, X_0, \bar{W}_0, W_0, \bar{T}_0, T_0\}$ to LMI conditions in (14), (27), (28), and set $k = 0$.

2) Solve the following LMI optimization problem for the variables

$$\begin{aligned} & \{\bar{X}, X, \bar{W}, W, \bar{T}, T\} \\ \min \quad & \mathrm{tr}(\bar{W}_k W + \bar{X}_k X + \bar{T}_k T + \bar{W} W_k + \bar{X} X_k + \bar{T} T_k) \\ \text{s.t.} \quad & \text{LMIs}(14), (27), (28) \end{aligned}$$

and set: $\bar{X}_{k+1} := \bar{X}$, $X_{k+1} := X$, $\bar{W}_{k+1} := \bar{W}$, $W_{k+1} := W$, $\bar{T}_T := \bar{T}$, $T_{k+1} := T$.

3) If $W \leq XT^{-1}X$, then set $\gamma_0 = \gamma$, $\tau_0 = \tau$ and return to step 1) after decreasing γ and increasing τ to some extent. Otherwise, set $k = k + 1$ and go to step 2) and repeat the procedure for a prescribed number of iterations, until finding W, X, T satisfying $W \leq XT^{-1}X$. If there exist no such matrices, then exit.

Remark 3. The main results in Theorem 1, matrix inequalities (13) and (14) rely on the non-zero eigenvalues λ_i of Laplacian matrix L. That means we must verify the matrix inequality for every non-zero eigenvalue of Laplacian matrix. However, the cone-complementary linearization algorithm will save us from the tedious works. The application of cone-complementary linearization algorithm coverts the matrix inequalities (13) and (14) which contain the nonlinear entries to LMIs (14), (27) and (28). Due to the convex property of LMIs, only two groups LMIs related to the smallest positve eigenvalue λ_2 and largest eigenvalue λ_n need to be verified. And this still works for matrix inequalities (29) and (30).

Theorem 2. Assume that the undirected graph $\mathcal{G}$ is connected. Given nonnegative constants τ and γ, the feedback control gain $K = LX^{-1}$ globally asymptotically stabilizes the closed-loop system (12) with an H_∞ distrubance attenuation level of γ, if there exist positive definite matrices X, $\bar{P}_{12}$, $\bar{P}_{22}$, $\bar{Z}_{11}$, $\bar{Z}_{12}$, $\bar{Z}_{22}$, $\bar{Z}_1$, Q, T and matrices L, N_1, N_2, N_3, N_4 with appropriate dimensions and scalar $\sigma > 0$ such that the following matrix inequalities are satisfied for $i = 2, \cdots, n$:

$$\begin{bmatrix} \widetilde{\Omega}_{i0} & \widetilde{\Xi}_{i1}^{\mathrm{T}} & \Xi_{i2}^{\mathrm{T}} & \Xi_{i3}^{\mathrm{T}} \\ * & \sigma\epsilon^2\lambda_{\mathcal{G}}^2 B_1 B_1^{\mathrm{T}} - T & 0 & 0 \\ * & * & -XT^{-1}X & 0 \\ * & * & * & -\sigma I \end{bmatrix} < 0 \tag{29}$$

$$\begin{bmatrix} X & \bar{P}_{12} \\ * & \bar{P}_{22} \end{bmatrix} > 0, \quad \begin{bmatrix} \bar{Z}_{11} & \bar{Z}_{12} \\ * & \bar{Z}_{22} - \bar{Z}_1 \end{bmatrix} > 0 \tag{30}$$

where $\widetilde{\Xi}_{i1} = [AX + \sigma\epsilon^2\lambda_{\mathcal{G}}^2 B_1 B_1^{\mathrm{T}} \ \lambda_i B_1 L \ B_2 \ 0 \ 0 \ 0 \ 0 \ 0], \Xi_{i3} = [0 \ L \ 0 \ 0 \ 0 \ 0 \ 0 \ 0]$ and $\widetilde{\Omega}_{i0}$ has same entries as Ω_{i0} except $\widetilde{\Omega}_{i11} = \Omega_{i11} + \sigma\epsilon^2\lambda_{\mathcal{G}}^2 B_1 B_1^{\mathrm{T}}$.

Proof. Replacing λ_i with $\lambda_i + \mu_i(t)$ in (13), we rewrite the matrix inequality conditions (13) as $\Sigma_i + \Theta_i\Pi_i + \Pi_i^{\mathrm{T}}\Theta_i^{\mathrm{T}} < 0$, where

$$\begin{aligned} \Theta_i^{\mathrm{T}} &= [\mu_i(t)B_1^{\mathrm{T}} \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ \mu_i(t)B_1^{\mathrm{T}} \ 0] \\ \Pi_i &= [0 \ L \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0] \end{aligned} \tag{31}$$

If there exists a $\sigma > 0$ such that $\Sigma_i + \sigma\Theta_i\Theta_i^{\mathrm{T}} + \sigma^{-1}\Pi_i^{\mathrm{T}}\Pi_i < 0$, we will get $\Sigma_i + \Theta_i\Pi_i + \Pi_i^{\mathrm{T}}\Theta_i^{\mathrm{T}} < 0$. Thus using Schur complement formula on $\Sigma_i + \sigma\Theta_i\Theta_i^{\mathrm{T}} + \sigma^{-1}\Pi_i^{\mathrm{T}}\Pi_i < 0$ results in (29). □

3 Numerical examples

In this section, we will give examples to show the effectiveness of our protocols. We consider a network with four agents, as shown in Fig.1, with the communication channel gains $a_{12}(t) = a_{21}(t) = 1 + \Delta_1(t)$, $a_{13}(t) = a_{31}(t) = 1 + \Delta_2(t)$, $a_{14}(t) = a_{41}(t) = 1 + \Delta_3(t)(|\Delta_i(t)| \leq 0.1)$.

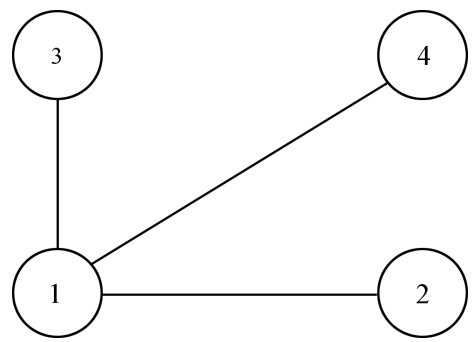

Fig. 1 Interaction undirected graph: $\mathcal{G}$

First, we consider the case of the uncertainties of $a_{ij}(t)$ equal 0, i.e. $\Delta_i(t) = 0$. Second, we give the results when $\Delta_i(t) \neq 0$. In simulations, the H_∞ performance index γ is chosen as 1, and communication delay $0 \leq d \leq \tau = 0.06$. Consider the open-loop system (1) with

$$A = \begin{bmatrix} 0 & -1 \\ 2 & 1 \end{bmatrix}, \ B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \ B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{32}$$

Table 1 Simulation results

$\Delta_i(t) = 0$	feasible	$K = \begin{bmatrix} -1.0716 & 1.0032 \\ -1.0014 & -1.0515 \end{bmatrix}$
$\Delta_i(t) \neq 0$	feasible	$K = \begin{bmatrix} -1.1548 & 0.8026 \\ -1.1134 & -1.7950 \end{bmatrix}$

4 Conclusions

In this paper, we design a state feedback protocol to solve the consensus control of multi-agent systems with external disturbances and model uncertainty on communication topology. An augmented type Lyapunov-Krasovskii functional is employed, and two consensus criteria in the form of matrix inequalities are derived which guarantee the consensus of multi-agent systems with input delay under the H_∞ controller. A cone complementary algorithm is used and the solution of H_∞ control problem is solved by using an iterative algorithm.

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