Sampled-data Consensus of Multi-agent Systems with General Linear Dynamics Based on a Continuous-time Model

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Abstract This paper discusses the sampled-data consensus problem of multi-agent systems with general linear dynamics and time-varying sampling intervals. To investigate the allowable upper bound of sampling intervals, we employ the property of discretization of sampled-data to identify the upper bound on the variable sampling intervals via a continuous-time model. Without considering the states in the sampling intervals, the decrease of Lyapunov function is guaranteed only at each sampling time. Consequently, it results in a more robust sampling interval which is obtained by verifying the feasibility of LMIs. Subsequently, provided a limited matrix variable, the control gain matrix K is solved by the LMI approach. Numerical simulations are provided to demonstrate the effectiveness of theoretical results.

Key words Multi-agent systems, sampled-data consensus, time-varying sampling intervals, Lyapunov theorem, continuous-time model

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Distributed coordination of multi-agent systems has attracted considerable attentions recently due to their extensive applications[1−2]. It is well known that consensus is a fundamental problem in distributed coordination of multi-agent systems and most of the work on consensus problems is based on ideal communication networks. As multi-agent systems are representative networked control systems, the communication networks cannot be considered ideally any more in reality[3−5], by which the information used to coordinate the collective behaviors among agents is transmitted. Owing to underlying communication imperfections and constraints, the effects of sampling, time delay, and quantization on consensus should be taken into account.

The sampled-data is the intrinsic property in digital controls which normally assume periodic samplings. However the case in the networked control is significantly different. Highly variable network conditions make the systems experience the network access delays and the transmission delays, which require system stability under time-varying sampling intervals as well. The main focus of our work is the sampled-data consensus problem. Due to the unreliability of communication channels, a more practical case than the continuous information transmission should be considered, that is agents may only be able to exchange information periodically but not continuously. Therefore the intermittent information transmission can be handled by the sampled-data formulation. Not merely to the case of intermittent information transmission, the sampled-data control in the coordination of multi-agents has been followed with interests in many practical situations, such as the limited data sensing, the networked transmission and so on.

Many significant results of the distributed coordination of multi-agent systems based on the sampled-data setting have been brought out lately. Reference [6] proposed a sampled-data algorithm to deal with intermittent information exchanges for double-integrator dynamics through average-like Lyapunov functions. Subsequently, Xie et al.[7−12] investigated the sampled-data consensus problems of first-order and second-order dynamics multi-agent systems under fixed and switching topologies, respectively. These results mainly ensured the consensus of the systems under the periodic samplings including the upper bounds of sampling period, some ranges of sampling period with gain coefficients and some necessary topology conditions. When the sampling interval is time varying, the aperiodic sampling in consensus problems has been considered in [13−17]. Reference [13] proposed a protocol based on the sampled-data control with time-varying sampling intervals and derived some conditions for consensus in the fixed topology case and the switching topology case, respectively. Reference [14] discussed the synchronous and asynchronous consensus problems via variable sampling intervals with time delays. Reference [15] dealt with the sampled-data consensus problem under the undirected fixed topology case, which gave the approach of how to choose the time-varying sampling interval. A novel protocol being applicable to the case with large sampling periods was proposed in [16]. This protocol was insensitive to the length of sampling periods because a closed-loop control system was guaranteed for the dynamics of each agent between any two sampling instants. Furthermore, sampled-data consensus problems with intrinsic nonlinear dynamics were discussed in [17].

Various approaches can be used for the sampled-data control with different formulations. For the sampled-data consensus problems, most of the existing results were obtained by the discrete-time approach[6−16, 18−20] as well as the input delay approach[14, 17, 21]. Comparing these two approaches, both of them have respective limitations in fact. We can notice that the exponential uncertainties of general linear discrete-time models, which are brought in by time-varying sampling intervals, are difficult to tackle and it is not easy to analyze the system when the time delays are larger than sampling intervals based on the discretization method. Likewise the conservative results are produced without considering the characteristics of time-varying delays modeled by the input delay approach. In addition, impulsive consensus algorithms were investigated

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in [22−23].

All the aforementioned literature focusing on sampled-data consensus of single and double integrator with time-varying sampling intervals, the objective of the present paper is to discuss the sampled-data consensus problem based on a continuous-time model, which exploits the property of discretization of sampled-data in the process of continuous-time evolution. The relationship between the discrete-time approach and the continuous-time approach is well elaborated by [24]. Different from the continuous-time approach modeled by an input delay, this paper is concerned with the sequence of samplings in the continuous-time model but not the states in the interval of any successive samplings. By this means, the obtained stability condition is looser than the case paying attention to the whole process of each sampling interval. It is shown that a less conservative condition is achieved compared with the existing results in [14, 17].

The remainder of this paper is organized as follows. Section 1 formulates the problem. The main results are stated in Section 2. Section 3 gives the simulation to validate the theoretical results. Conclusions are given in Section 4.

Notations. The sets $\mathbf{R}^+, \mathbf{R}^*, \mathbf{R}^{m\times n}, \mathbf{S}^n$ denote the nonnegative scalars, $*$-dimensional vectors, $m\times n$ matrices, and symmetric matrices of $\mathbf{R}^{n\times n}$ throughout the paper, respectively. $\mathbf{K}$ is defined as the set of differentiable functions from an interval of $[0,\mathcal{T}]$ to $\mathbf{R}^n$, where $\mathcal{T}\in\mathbf{R}^+$. The notation $\|\cdot\|$ denotes the Euclidean norm, the superscript 'T' stands for matrix transposition, and $P>0$ means that P is symmetric and positive definite for all $P\in\mathbf{S}^n$. For any matrix $A\in\mathbf{R}^{n\times n}$, the notation $\mathrm{He}\{A\}$ refers to $A+A^{\mathrm{T}}$. The symbols I_* and 0_* represent the identity and the zero matrices of $*$-dimensions.

1 Problem formulation

This paper addresses the sampled-data consensus problems where the sampling interval is time-varying, and the dynamics of the i-th agent is described by the general linear time-invariant differential equation

$$\dot{x}_i(t) = Ax_i(t) + Bu_i(t), i=1,\cdots,N, t\geq t_0 \tag{1}$$

where $x_i\in\mathbf{R}^n$ is the state vector, $u_i\in\mathbf{R}^m$ is the input vector. A and B are real constant matrices with appropriate dimensions and the following assumption holds:

Assumption 1. (A,B) is stabilizable and not all the eigenvalues of the state matrix A are in the open left half plane.

We use a directed graph $G=(V,E)$ to model the network topology of multi-agent systems, where $V=\{v_1,v_2,\cdots,v_N\}$ is the node set, $E\subseteq V\times V$ is the edge set. An edge $e_{ij}\in E$ in G denotes that there is a directed information path from agent j to agent i. $Adj=[a_{ij}]\in\mathbf{R}^{N\times N}$ is the weighted adjacency matrix associated with G such that $a_{ij}>0$ if $e_{ij}\subseteq V\times V$, and $a_{ij}=0$ otherwise, as well $a_{ii}=0$ for all $i=1,\cdots,N$. The neighbor set of the i-th agent is denoted by $N_i(t)=\{j|a_{ij}>0, i\neq j\}, i\in V$. $L=[l_{ij}]\in\mathbf{R}^{N\times N}$ is Laplacian matrix, where $l_{ij}=-a_{ij}, l_{ii}=\sum_{j=1}^N a_{ij}, j\neq i$.

We say that multi-agent systems (1) solve a consensus problem asymptotically under given $u_i(t), i=1,\cdots,N$ if for any initial states and any $i,j=1,\cdots,N, \lim_{t\to\infty}\|x_i(t)-x_j(t)\|=0$. When the information transmissions among all agents are continuous, a general consensus protocol of the i-th agent is of the following form:

$$u_i(t) = K\sum_{j\in N_i(t)} a_{ij}(x_j(t)-x_i(t)), i=1,\cdots,N, t\geq 0 \tag{2}$$

where K is the feedback gain matrix.

This paper is interested in the sampled-data consensus problem based on the suggested protocol (2), that is the state information relative to its neighbors can only be obtained by each agent at discontinuous time. It is assumed that each agent can measure its states relative to its neighbors by special sensors at discrete time and the measurements are sent to its controller to form a control signal, which will be held by a zero-order-hold until a new control signal is updated. Samplers in sensors are clock-driven, whereas controllers and zero-order-holds are event-driven. We assume that the sampling times of all agents are synchronous so that they can be denoted by $s_0, s_1, \cdots$. The update time of agent i at time k is defined as $t_k^{(i)}$. When we do not consider delays induced by networks the update times of all agents are synchronous, that is $t_k^{(1)}=\cdots=t_k^{(N)}, k=0,1,\cdots$. Due to the computation and communication constraints, sampling intervals are subject to time variation and uncertainty. We assume that

$$0<T_k=t_{k+1}-t_k\leq T_{\max}, k=0,1,\cdots \tag{3}$$

where T_k is the time-varying sampling interval, $T_{\max}$ denotes the maximum sampling interval. Based on the above assumptions, the following control law is proposed

$$\begin{aligned} u_i(t) &= K\sum_{j\in N_i} a_{ij}(x_j(t_k)-x_i(t_k)), \\ i &= 1,\cdots,N, t\in[t_k,t_{k+1}), k=0,1,\cdots \end{aligned} \tag{4}$$

where $K\in\mathbf{R}^{m\times n}$ is the control gain matrix to be designed.

The system (1) combined with (4) can be written as the following compact form:

$$\begin{aligned} \dot{x}(t) &= (I_N\otimes A)x(t) - (L\otimes(BK))x(t_k), \\ & t\in[t_k,t_{k+1}), k=0,1,\cdots \end{aligned} \tag{5}$$

where $x(t)=[x_1^{\mathrm{T}}(t),\cdots,x_N^{\mathrm{T}}(t)]^{\mathrm{T}}$. Our objective is to derive sufficient conditions for asymptotical consensus of (5) depending on K, T and the network topology.

2 Main results

In this section, we will provide some sufficient conditions which solve the sampled-data consensus problem for system (5) under the fixed directed topology.

2.1 Analysis of sampled-data consensus

The following assumption is made:

Assumption 2. The graph G is directed and contains a spanning tree.

Let $r=[r_1,\cdots,r_N]^{\mathrm{T}}\in\mathbf{R}^N$ be the left eigenvector of L associated with the zero eigenvalue, satisfying $r^{\mathrm{T}}\mathbf{1}_N=1$, then introduce the following error variable:

$$\delta(t) = x(t) - ((\mathbf{1}_N r^{\mathrm{T}})\otimes I_n)x(t) \tag{6}$$

which satisfies $(r^{\mathrm{T}}\otimes I_n)\delta(t)=0_n$.

According to (6), the consensus problem of (5) is equivalent to the asymptotic stability problem of the error dynamics (7).

$$\dot{\delta}(t) = (I_N \otimes A)\delta(t) - (L \otimes (BK))\delta(t_k), \quad t \in [t_k, t_{k+1}), k = 0, 1, \cdots \tag{7}$$

Let nonsingular matrix $T \in \mathbf{R}^{N\times N}, W \in \mathbf{R}^{N\times(N-1)}$ and $\Delta \in \mathbf{R}^{(N-1)\times(N-1)}$ be such that

$$T = \begin{bmatrix} \mathbf{1}_N & W \end{bmatrix}, T^{-1}LT = J = \begin{bmatrix} 0 & 0 \\ 0 & \Delta \end{bmatrix} \tag{8}$$

where the diagonal entries of $\triangle$ are the nonzero eigenvalues of L. With the state transformation $\xi(t) = (T^{-1} \otimes I_n)\delta(t)$, (7) can be represented as following:

$$\dot{\xi}(t) = (I_N \otimes A)\xi(t) - (J \otimes (BK))\xi(t_k), \quad t \in [t_k, t_{k+1}), k = 0, 1, \cdots \tag{9}$$

where $\xi(t) = [\xi_1, \chi(t)]^{\mathrm{T}}, \chi(t) = [\xi_2^{\mathrm{T}}, \cdots, \xi_N^{\mathrm{T}}]^{\mathrm{T}}$. As $\xi_1 = (r^{\mathrm{T}} \otimes I_n)\delta(t) \equiv 0_n$, the error dynamics (9) is asymptotically stable if and only if the following system is asymptotically stable

$$\dot{\chi}(t) = (I_{N-1} \otimes A)\chi(t) - (\Delta \otimes (BK))\chi(t_k), \quad t \in [t_k, t_{k+1}), k = 0, 1, \cdots \tag{10}$$

Note that the system (10) is piecewise continuous, or rather, the problem of sampled-data systems possesses the characteristics of discrete-time and continuous-time systems. Motivated by the relation between discrete-time and continuous-time systems in sampled-data control which is revealed by [24], we deal with the sampled-data consensus problem based on the continuous-time model. Define the function $\chi_k(\tau) : \mathbf{K}$ such that $\chi_k(\tau) = \chi(t_k + \tau)$ for all $\tau \in [0, T_k]$ and any nonnegative integer k. Then the following lemma is introduced, which plays an important role in the proof of our main theorems.

Lemma 1[24]**.** Let $0 < T_1 \leq T_2$ be two positive scalars and $V : \mathbf{R}^n \to \mathbf{R}^+$ be a differentiable function for which there exist positive scalars $\mu_1 < \mu_2$ and p such that

$$\mu_1 \|\chi\|^p \leq V(\chi) \leq \mu_2 \|\chi\|^p, \quad \forall \chi \in \mathbf{R}^n \tag{11}$$

If there exists a continuous and differentiable functional $V_0 : [0, T_2] \times \mathbf{K} \to \mathbf{R}$ which satisfies for all $\chi_k \in \mathbf{K}$

$$V_0(T_k, \chi_k) = V_0(0, \chi_k), \quad \forall T_k \in [T_1, T_2] \tag{12}$$

and such that, for all $(k, T_k, \tau) \in \mathbf{N} \times [T_1, T_2] \times [0, T_k]$,

$$\frac{\mathrm{d}}{\mathrm{d}\tau}(V(\chi_k(\tau)) + V_0(\tau, \chi_k)) < 0 \tag{13}$$

then the system (10) is asymptotically stable.

Lemma 1 shows that the sampled-data system (10) is asymptotically stable, however, the differentiation of the positive definite function $V(\chi)$ on τ in (10) is not necessarily negative definite all the time. That is, (12) and (13) are equivalent to the discrete-time Lyapunov theorem, where the states are just considered at the sampling times. It means that the stability condition is less conservative than the input delay approach based on the continuous-time mode.

Theorem 1. Suppose that Assumptions 1 and 2 hold, and $A - \lambda_i BK$ is Hurwitz for all λ_i, where $\lambda_i, i = 2, \cdots, N$ are the nonzero eigenvalues of L. If there exist $P > 0$, $R > 0, Q_1 \in \mathbf{S}^{(N-1)n}, Q_2 \in \mathbf{R}^{(N-1)n\times(N-1)n}, X \in \mathbf{R}^{4(N-1)n\times 2(N-1)n}$ that satisfy the following LMIs:

$$\Xi_1 + XS_1 + S_1^{\mathrm{T}} X^{\mathrm{T}} < 0 \tag{14}$$

$$\Xi_2 + XS_2 + S_2^{\mathrm{T}} X^{\mathrm{T}} < 0 \tag{15}$$

where

$$\Xi_1 = \mathrm{He}\{M_1^{\mathrm{T}} P M_2 - M_{23}^{\mathrm{T}}(\frac{1}{2}Q_1 M_{23} + Q_2 M_3)\} + T_{\max}\mathrm{He}\{M_1^{\mathrm{T}}(\frac{1}{2}RM_1 + Q_1 M_{23} + Q_2 M_3)\}$$

$$\Xi_2 = \mathrm{He}\{M_1^{\mathrm{T}} P M_2 - M_{23}^{\mathrm{T}}(\frac{1}{2}Q_1 M_{23} + Q_2 M_3)\} - T_{\max} M_4^{\mathrm{T}} R M_4$$

$$S_1 = \begin{bmatrix} -I_{(N-1)n} & C_0 + C_1 & -C_1 & \mathbf{0}_{(N-1)n} \\ \mathbf{0}_{(N-1)n} & I_{(N-1)n} & -I_{(N-1)n} & -0I_{(N-1)n} \end{bmatrix}$$

$$S_2 = \begin{bmatrix} -I_{(N-1)n} & C_0 + C_1 & -C_1 & \mathbf{0}_{(N-1)n} \\ \mathbf{0}_{(N-1)n} & I_{(N-1)n} & -I_{2(N-1)} & -T_{\max} I_{(N-1)n} \end{bmatrix}$$

with $M_1 = [I\ 0\ 0\ 0], M_2 = [0\ I\ 0\ 0], M_3 = [0\ 0\ I\ 0], M_4 = [0\ 0\ 0\ I], M_{23} = M_2 - M_3, C_0 = I_{N-1} \otimes A - \Delta \otimes (BK), C_1 = \Delta \otimes (BK)$.

Then the consensus problem of system (1) is solved by (4) for any time-varying sampling interval $T_k \in (0, T_{\max}], k = 0, 1, \cdots$, where $T_{\max}$ is the allowable upper bound.

Proof. According to the definition of $\chi_k(\tau)$, the system (10) can be rewritten as

$$\dot{\chi}_k(\tau) = C_0 \chi_k(\tau) + C_1(\chi_k(\tau) - \chi_k(0)) \quad \tau \in [0, T_k], k = 0, 1, \cdots \tag{16}$$

Define a quadratic Lyapunov function $V(\chi_k) = \chi_k^{\mathrm{T}} P \chi_k$ for all χ_k in (16), which obviously satisfies (11). A continuous and differentiable functional V_0 satisfying (12) is defined by

$$V_0(\tau, \chi_k) = (T_k - \tau)\Big\{ \int_0^\tau \dot{\chi}_k^{\mathrm{T}}(s) R \dot{\chi}_k(s) \mathrm{d}s + (\chi_k(\tau) - \chi_k(0))^{\mathrm{T}}[Q_1(\chi_k(\tau) - \chi_k(0)) + Q_2 \chi_k(0)] \Big\} \tag{17}$$

Then the differentiation of $V + V_0$ on τ along the solution of (16) is given and bounded by $-\int_0^\tau \dot{\chi}_k^{\mathrm{T}}(s) R \dot{\chi}_k(s) \mathrm{d}s \leq -\tau \upsilon(\tau)^{\mathrm{T}} R \upsilon(\tau)$, where $\upsilon(\tau) = \frac{1}{\tau}\int_0^\tau \dot{\chi}_k(s)\mathrm{d}s, \lim_{\tau\to 0} \upsilon(\tau) = \dot{\chi}_k(0)$.

$$\frac{\mathrm{d}}{\mathrm{d}\tau}(V(\chi_k(\tau)) + V_0(\tau, \chi_k)) \leq \eta_k^{\mathrm{T}}(\tau) \Xi \eta_k(\tau) \quad S\eta_k(\tau) = 0 \tag{18}$$

where

$$\Xi = \mathrm{He}\left\{ M_1^{\mathrm{T}} P M_2 - M_{23}^{\mathrm{T}} \left(\frac{1}{2} Q_1 M_{23} + Q_2 M_3 \right) \right\} + (T_k - \tau)\mathrm{He}\left\{ M_1^{\mathrm{T}} \left(\frac{1}{2} R M_1 + Q_1 M_{23} + Q_2 M_3 \right) \right\} - \tau M_4^{\mathrm{T}} R M_4$$

$$S = \begin{bmatrix} -I_{(N-1)n} & C_0 + C_1 & -C_1 & \mathbf{0}_{(N-1)n} \\ \mathbf{0}_{(N-1)n} & I_{(N-1)n} & -I_{(N-1)n} & -\tau I_{(N-1)n} \end{bmatrix}$$

$$\eta_k(\tau) = [\dot{\chi}_k^{\mathrm{T}}(\tau), \chi_k^{\mathrm{T}}(\tau), \chi_k^{\mathrm{T}}(0), \upsilon(\tau)^{\mathrm{T}}]^{\mathrm{T}}$$

According to Finsler Lemma[25], $\frac{\mathrm{d}}{\mathrm{d}\tau}(V + V_0) < 0$ if and only if $\Xi + XS + S^{\mathrm{T}} X^{\mathrm{T}} < 0$. Employing the convexity and linearity property of τ and T_k in the above inequality,

$\frac{\mathrm{d}}{\mathrm{d}\tau}(V+V_0)<0$ for all $\tau\in[0,T_k]$ and $T_k\in(0,T_{\max}]$ if and only if (14) and (15) hold.

When $T_{\max}$ tends to zero, (14) and (15) both tend to

$$\begin{aligned}
&\mathrm{He}\left\{M_1^{\mathrm{T}}PM_2-M_{23}^{\mathrm{T}}\left(\frac{1}{2}Q_1M_{23}+Q_2M_3\right)\right\}+\\
&XS_1+S_1^{\mathrm{T}}X^{\mathrm{T}}<0\\
&\Leftrightarrow {S_1^{\perp}}^{\mathrm{T}}\mathrm{He}\Big\{M_1^{\mathrm{T}}PM_2-M_{23}^{\mathrm{T}}\left(\frac{1}{2}Q_1M_{23}+Q_2M_3\right)\Big\}S_1^{\perp}<0\\
&\Leftrightarrow PC_0+C_0^{\mathrm{T}}P<0\\
&\Leftrightarrow \tilde{P}(A-\lambda_iBK)+(A-\lambda_iBK)^{\mathrm{T}}\tilde{P}<0, i=2,\cdots,N^{[26]}
\end{aligned}$$

where the second inequality is derived from Finsler lemma, $S_1^{\perp}$ is the orthocomplement of S_1 and $\tilde{P}>0$. Thus (14) and (15) are feasible. This leads to the asymptotical stability of the system (10) according to Lemma 1. □

Remark 1. As can be seen from the proof of Theorem 1, the discretization is avoided, meanwhile, the decrease of $V(\chi)$ is just required at $\tau=0$ or $\tau=T_k$. It weakens the requirement of decrease of Lyapunov function all the time, hence the reduction in the conservatism of the allowable maximum sampling intervals is reasonable.

Remark 2. It is indicated by Theorem 1 that the control gain matrix K, by which (2) solves the continuous-time consensus of (1), can be used for sampled-data consensus, provided that the sampling intervals are small enough. By virtue of this, the design of sampled-data controller can be obtained directly from existing continuous-time controller, nevertheless, continuous-time control gain matrix K is incapable of supporting the allowable upper bound controller of time-varying sampling intervals. An LMI approach is proposed to design the controller for the allowable upper bound in the next subsection.

2.2 Controller design

Matrix inequalities of Theorem 1 are nonlinear as K is variable matrix. Applying Finsler Lemma leads to that

$$\Xi_3+\tilde{X}S_3+S_3^{\mathrm{T}}\tilde{X}^{\mathrm{T}}<0 \tag{19}$$

$$\Xi_4+\tilde{X}S_4+S_4^{\mathrm{T}}\tilde{X}^{\mathrm{T}}<0 \tag{20}$$

where

$$\begin{aligned}
\Xi_3=\,&\mathrm{He}\left\{\tilde{M}_1^{\mathrm{T}}P\tilde{M}_2-\tilde{M}_{23}^{\mathrm{T}}\left(\frac{1}{2}Q_1\tilde{M}_{23}+Q_2\tilde{M}_3\right)\right\}+\\
&T_{\max}\mathrm{He}\left\{\tilde{M}_1^{\mathrm{T}}\left(\frac{1}{2}R\tilde{M}_1+Q_1\tilde{M}_{23}+Q_2\tilde{M}_3\right)\right\}\\
\Xi_4=\,&\mathrm{He}\left\{\tilde{M}_1^{\mathrm{T}}P\tilde{M}_2-\tilde{M}_{23}^{\mathrm{T}}\left(\frac{1}{2}Q_1\tilde{M}_{23}+Q_2\tilde{M}_3\right)\right\}-\\
&T_{\max}\tilde{M}_{23}^{\mathrm{T}}R\tilde{M}_{23}\\
S_3=\,&\left[\begin{array}{ccc}-I_{(N-1)n} & \frac{1}{2}C_0 & \frac{1}{2}C_0\end{array}\right]\\
S_4=\,&\left[\begin{array}{ccc}-I_{(N-1)n} & C_0+C_1 & -C_1\end{array}\right]
\end{aligned}$$

with $\tilde{M}_1=[I\ 0\ 0], \tilde{M}_2=[0\ I\ 0], \tilde{M}_3=[0\ 0\ I], \tilde{M}_{23}=\tilde{M}_2-\tilde{M}_3$, and $P>0, R>0, Q_1\in\mathbf{S}^{(N-1)n}, Q_2\in\mathbf{R}^{(N-1)n\times(N-1)n}, \tilde{X}\in\mathbf{R}^{3(N-1)n\times(N-1)n}$.

Let $\tilde{X}^{\mathrm{T}}=[I_{N-1}\otimes Y, I_{N-1}\otimes Y, I_{N-1}\otimes Y]$, where $Y\in\mathbf{R}^{n\times n}$ is nonsingular, and $\tilde{\Delta}=\mathrm{diag}\{(I_{N-1}\otimes Y)^{-1}, (I_{N-1}\otimes Y)^{-1}, (I_{N-1}\otimes Y)^{-1}\}$. We multiply (19), (20) by $\tilde{\Delta}$ and $\tilde{\Delta}^{\mathrm{T}}$ on the left and the right, respectively. Denoting $V=(I_{N-1}\otimes Y)^{-1}, W_1=VPV,\ W_2=VQ_1V,\ W_3=VQ_2V,\ W_4=VRV,\ Z=KY^{-1}$, we obtain the following LMIs:

$$\Xi_5+\bar{X}S_5+S_5^{\mathrm{T}}\bar{X}^{\mathrm{T}}<0 \tag{21}$$

$$\Xi_6+\bar{X}S_6+S_6^{\mathrm{T}}\bar{X}^{\mathrm{T}}<0 \tag{22}$$

where

$$\begin{aligned}
\Xi_5=\,&\mathrm{He}\left\{\tilde{M}_1^{\mathrm{T}}W_1\tilde{M}_2-\tilde{M}_{23}^{\mathrm{T}}\left(\frac{1}{2}W_2\tilde{M}_{23}+W_3\tilde{M}_3\right)\right\}+\\
&T_{\max}\mathrm{He}\left\{\tilde{M}_1^{\mathrm{T}}\left(\frac{1}{2}W_4\tilde{M}_1+W_2\tilde{M}_{23}+W_3\tilde{M}_3\right)\right\}\\
\Xi_6=\,&\mathrm{He}\left\{\tilde{M}_1^{\mathrm{T}}W_1\tilde{M}_2-\tilde{M}_{23}^{\mathrm{T}}\left(\frac{1}{2}W_2\tilde{M}_{23}+W_3\tilde{M}_3\right)\right\}-\\
&T_{\max}\tilde{M}_{23}^{\mathrm{T}}W_4\tilde{M}_{23}\\
S_5=\,&\left[\begin{array}{ccc}-V & \frac{1}{2}\bar{C}_0 & \frac{1}{2}\bar{C}_0\end{array}\right]\\
S_6=\,&\left[\begin{array}{ccc}-V & \bar{C}_0+\bar{C}_1 & -\bar{C}_1\end{array}\right]\\
\bar{X}^{\mathrm{T}}=\,&\left[\begin{array}{ccc}I_{(N-1)n} & I_{(N-1)n} & I_{(N-1)n}\end{array}\right]\\
\bar{C}_0=\,&(I_{N-1}\otimes A)V-\Delta\otimes(BZ), \bar{C}_1=\Delta\otimes(BZ)
\end{aligned}$$

Based on the above LMIs (21) and (22), we can get the following algorithm to solve the control gain matrix K.

Algorithm 1. Given (A,B) that is stabilizable and G containing a directed spanning tree, a controller in the form of (4) can be constructed according the following steps to solve the sampled-data consensus problem of (1) under time-varying sampling intervals.

Step 1. Choose a small enough initial $T_{\max}=T_0$ so that (21) and (22) are feasible.

Step 2. Choose an appropriate δ and let $T_{\max}=T_0+\delta$, then verify the feasibility of (21) and (22). If they are feasible, then go to next step, or else stop and $T_{\max}=T_0$.

Step *k*. Let $T_{\max}=T_0+(k-1)\delta$, and verify the feasibility of (21) and (22). If they are feasible, then go to next step, or else stop and $T_{\max}=T_0+(k-2)\delta$. Considering the control input should be bounded, we can bound $\|K\|\le\sigma$, where $\sigma>0$ is constant. Hereby, the maximum $T_{\max}$ will be found after finite steps. Corresponding to the allowable maximum $T_{\max}$, matrix variables V and Z are solved. Consequently, $K=ZY^{-1}$, where Y is extracted from V^{-1}.

Remark 3. The control gain matrix K is solved by the above LMIs, where the degree of freedom of $\tilde{X}$ is reduced. Obviously, if we let $\tilde{X}^{\mathrm{T}}=[I_{N-1}\otimes Y_1, I_{N-1}\otimes Y_2, I_{N-1}\otimes Y_3]$, the result will be less conservative while the number of matrix variables is increased.

3 Numerical examples

In this section, theoretical results will be verified by numerical simulations. Consider a second-order multi-agent network of six agents, whose dynamics are specified by A, B.

$$A=\left[\begin{array}{cc}0 & 1\\ 0 & 0\end{array}\right], B=\left[\begin{array}{c}0\\ 1\end{array}\right] \tag{23}$$

The initial states of the agents are $X_1(t_0)=[60,\ 80]^{\mathrm{T}}, X_2(t_0)=[30,\ 60]^{\mathrm{T}}, X_3(t_0)=[0,\ 0]^{\mathrm{T}}, X_4(t_0)=[-10,\ -5]^{\mathrm{T}}, X_5(t_0)=[-30,\ -10]^{\mathrm{T}}, X_6(t_0)=[-120,\ -15]^{\mathrm{T}}$, respectively. In order to reveal the progress of the allowable upper bound of the variable sampling intervals, we compare Theorem 1 with the corresponding results in [14, 17], which are obtained by the discrete-time approach and the input delay approach, respectively. In order to provide the same conditions among these approaches, the fixed network topology in Fig. 1 is chosen, and we let

$$K = \begin{bmatrix} 1 & \alpha \end{bmatrix}$$

According to the network topology in Fig. 1, we get $\alpha > 1.7363$. The choice of this gain coefficient in [14] is dependent on the network topology. It should be noticed that the maximum sampling interval in [14] is not only dependent on the gain coefficient and network topology but also another variable. The variable is denoted as 'α' in [14] but it is not the same as the gain coefficient here. The sampling interval reaches the upper bound when the variable is set as 0.3 in [14]. To compare Theorem 1 with the results in [14] and [17], we calculate the allowable upper bounds with a group of gain coefficients.

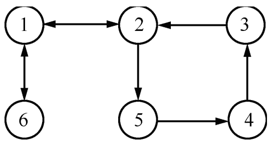

Fig. 1 Network topology

From Table 1, we can see that the upper bounds in different methods are all decreasing by the increase of the gain coefficient. The result obtained in [17] by the input delay approach is the most conservative, whereas the result in Theorem 1 is the least conservative.

For the case of $\alpha = 2$ and $T_{\max} = 0.3050$, T_k is selected from (0, $T_{\max}$] stochastically and the states of all agents obtained by the protocol (4) are shown in Fig. 2. The results verify the theoretical analysis very well. As shown in Fig. 3, the consensus cannot be achieved because the time-varying sampling intervals exceed the upper allowable bound. The convergence rate of consensus decreases as α increases, which can be seen from Fig. 2 and Fig. 4.

Table 1 The upper bound on the variable samplings

Gain╲Method	[14]	[17]	Theorem 1
α=2	0.2070	0.1080	0.3050
α=3	0.1500	0.0720	0.2040
α=4	0.1150	0.0540	0.1530
α=5	0.0930	0.0440	0.1230
α=7	0.0670	0.0310	0.0880
α=12	0.0400	0.0180	0.0510
α=17	0.0280	0.0130	0.0360
α=22	0.0220	0.0100	0.0280

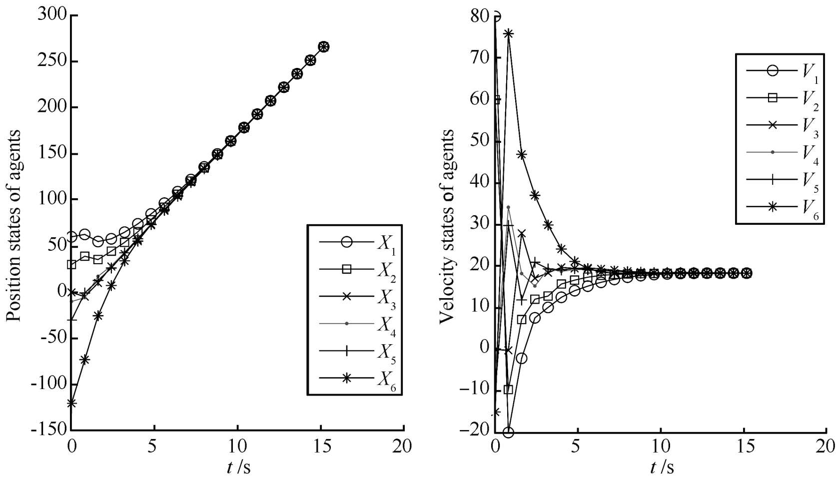

Fig. 2 The evolution of states of each agent with $\alpha = 2$ and $T_k \in (0, 0.3050]$

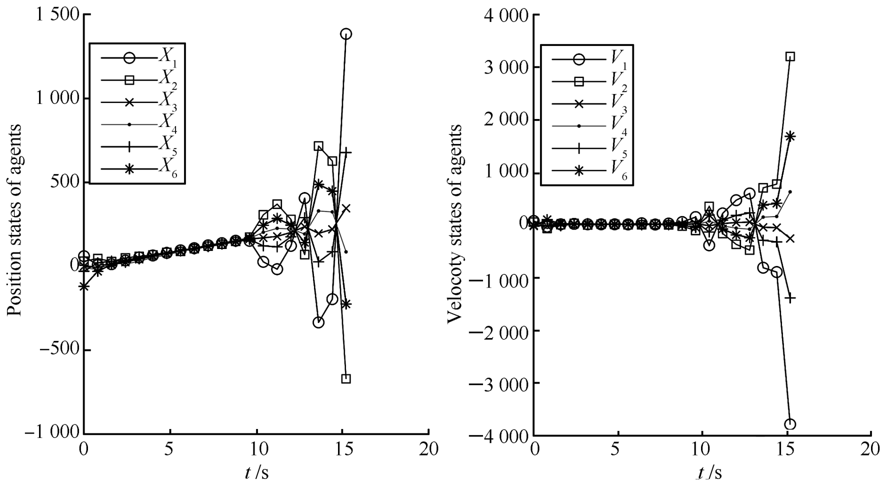

Fig. 3 The evolution of states of each agent with $\alpha = 2$ and $\max_{k=0,1,\cdots} T_k > 0.3050$

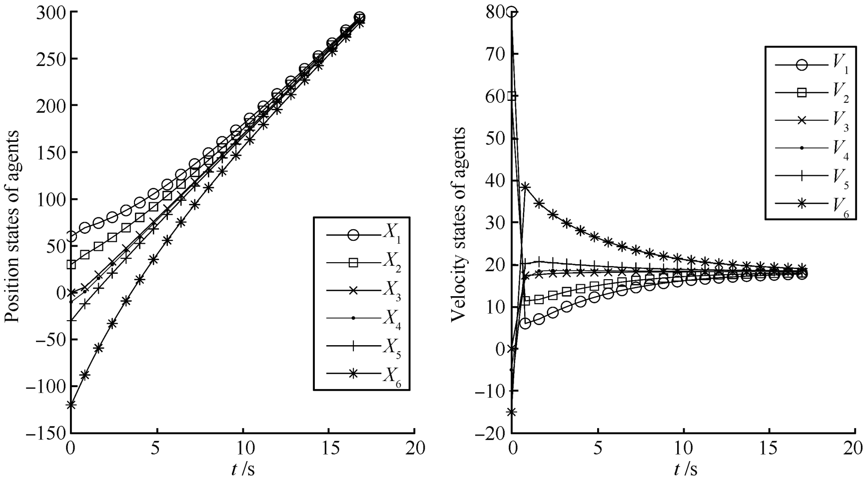

Fig. 4 The evolution of states of each agent with $\alpha = 5$ and $T_k \in (0, 0.1230]$

4 Conclusion

This paper studies the sampled-data consensus of multi-agent systems under the fixed directed network topology. Different from a general discrete approach, this paper mainly utilizes the inherent property of discretization in sampled-data control via a continuous-time model. It not only avoids the difficulty of exponential uncertainty due to the time-varying sampling intervals, but also the result is less conservative than the input delay approach even if the characteristic of delay impulsive system is exploited. In addition, it can be extended to the case of time-varying sampling intervals with diverse input delays, where handling of the asynchronous update times among all agents is the key issue.

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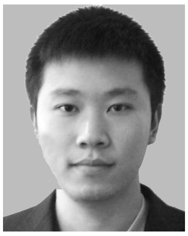

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