Statistical χ^2 Testing Based Fault Detection for Linear Discrete Time-delay Systems

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Abstract This paper is concerned with statistical χ^2 testing based fault detection (FD) for a class of linear discrete time-varying (LDTV) stochastic systems with delayed state. Different from the traditional residual based FD, we propose to construct the evaluation function by directly using measurement observations. Then an equivalent solution can be given in terms of Riccati recursion by utilizing projection and innovation analysis technique. Moreover, the fault free case evaluation function is with central χ^2 distribution and the heavy computational burden is reduced. Furthermore, strategies of χ^2 statistic testing on evaluation function are also discussed. Finally, a numerical example is given to illustrate the proposed method.

Key words Fault detection (FD), linear discrete time-delay system, evaluation function, χ^2 distribution, statistical testing

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Research on model-based fault detection and isolation (FDI) has received much attention during the past three decades; see for example [1−10] and references therein. Generally speaking, model-based FD consists of two stages, i.e., the residual generation and the residual evaluation. As summarized in [1−2], the first and important task is to design a fault detection filter (FDF) by maximizing the sensitivity of residual to fault and simultaneously minimizing the robustness of residual to unknown input. Then a fault indicator can be delivered by comparing a chosen residual evaluation function with a prescribed threshold.

Under the assumption of unknown input being Gaussian noise, the well known Kalman filter is a suitable choice of residual generator. Then a statistical testing of whiteness, mean and covariance of the residual can be used to detect the occurrence of a fault[3]. In [11], a least square based residual generation and matching approach was proposed for a class of discrete time-varying networked sensing systems with incomplete measurements.

For linear systems subject to l_2-norm bounded unknown input, much attention has been paid to residual generation in the framework of H_∞ filtering or H_∞ optimization. The former one was to formulate the design of FDF as an standard H_∞ filtering in the sense of minimizing the $\mathcal{L}_2$-induced gain from the unknown input to the error between the residual and the fault[12−16]. The another one was to define the sensitivity and robustness as an $\mathcal{H}_-$ index or H_∞ norm and formulated the underlying FD problem as an H_∞ optimization; see e.g. [17−25]. In [18], a unified solution was proposed to the H_∞/H_∞ and/or H_-/H_∞ optimization problem and, through co-inner-outer factorization, a residual generator was given in terms of Riccati equation. In [19] and [24], optimal solutions of H_-/H_∞ and H_∞/H_∞ were obtained for LDTV systems. An Krein space approach was proposed for robust H_∞ FD and a solution of FDF was given in terms of Riccati recursion[25].

On the other hand, time-delays are frequently encountered in practical control systems and efforts have been paid to the investigation of H_∞ FD for time-delay systems[26−31]. In [27] and [30−31], the problem of FD, estimation, and compensation for time-delay systems were dealt with. Reference [32] dealt with the problem of finite horizon H_∞ fault estimation for linear discrete time-varying (LDTV) systems with delayed states by using linear projection in Krein space, while [33] concerned only the delay of measurement output. It should be pointed out that the existing results of FD for linear time-delay systems concern with norm bounded unknown input and most of the focuses are on residual generation. The problem of FD for networked systems subject to time-delays was investigated in [34−38]. By considering the main purpose of FD, i.e., delivering a fault indicator after its occurrence, a nature idea is to generate evaluation function directly in the first stage and perform FD as statistical testing in the second stage, and this motivates the present study.

This paper is concerned with FD for a class of LDTV stochastic systems with delayed state. Different from traditional residual based FD, we will first consider to directly generate an evaluation function by using a quadratic form of measurement observations. Under the assumption of unknown input being jointly normal distributed, the evaluation function will be proved to be central χ^2 distribution and, by using projection and innovation analysis, the evaluation function for LDTV systems with delayed state will be given in terms of Riccati recursion. Similar to the statistical testing of residual, the χ^2 statistical testing of evaluation function will also be discussed for detecting the occurrence of a fault. A numerical example will be given to show the effectiveness of the proposed method.

Notation. Through out the paper, $\mathcal{E}[\cdot]$ denotes expectation of $[\cdot]$. Superscripts "−1" and "T" stand for the inverse and transpose of a matrix, respectively. $\mathbf{R}^{n\times m}$ is the set of all $n\times m$ real matrices. I is the identity matrix with appropriate dimensions. For a real symmetric matrix P, $P>0$ (respectively, $P<0$) means that P is a real positive definite (respectively, negative definite) matrix. diag$\{\cdot\}$ denotes a block-diagonal matrix. $\mathcal{L}\{\cdot\}$ denotes the linear space spanned by sequence $\{\cdot\}$. For zero mean stochastic vectors α and β, $\langle\alpha,\beta\rangle$ stands for the covariance

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matrix and $R_\alpha = \langle \alpha, \alpha \rangle$.

1 Problem formulation

Consider the following LDTV system with delayed state

$$\begin{cases} x(k+1) = A(k)x(k) + A_\tau(k)x(k-\tau) + \\ \qquad\qquad B_d(k)d(k) + B_f(k)f(k) \\ y(k) = C(k)x(k) + v(k) + D_f(k)f(k) \\ x(0) = x_0, \ \ x(k) = 0, \quad \text{for } k < 0 \end{cases} \tag{1}$$

where $x(k) \in \mathbf{R}^n$, $y(k) \in \mathbf{R}^m$, $d(k) \in \mathbf{R}^q$, $v(k) \in \mathbf{R}^m$ and $f(k) \in \mathbf{R}^p$ are the state, measurement output, process noise, measurement noise and fault to be detected, respectively; $\tau > 0$ is the state delay which is an integer; $f(k) \in l_2\,[0, N]$; $A(k)$, $A_\tau(k)$, $B_d(k)$, $B_f(k)$, $C(k)$ and $D_f(k)$ are known matrices with appropriate dimensions. The initial state x_0 and $d(k)$, $v(k)$ are assumed to be zero mean and normally distributed vectors with

$$\left\langle \begin{bmatrix} x_0 \\ d(k) \\ v(k) \end{bmatrix}, \begin{bmatrix} x_0 \\ d(j) \\ v(j) \end{bmatrix} \right\rangle = \text{diag}\{P_0, \ R(k)\delta_{kj}, \ Q(k)\delta_{kj}\} \tag{2}$$

where $\delta_{kj} = I$ for $k = j$, $\delta_{kj} = 0$ for $k \neq j$.

Define

$$f_N = \begin{bmatrix} f^{\mathrm{T}}(0) & f^{\mathrm{T}}(1) & \cdots & f^{\mathrm{T}}(N) \end{bmatrix}^{\mathrm{T}}$$
$$d_N = \begin{bmatrix} d^{\mathrm{T}}(0) & d^{\mathrm{T}}(1) & \cdots & d^{\mathrm{T}}(N) \end{bmatrix}^{\mathrm{T}}$$
$$v_N = \begin{bmatrix} v^{\mathrm{T}}(0) & v^{\mathrm{T}}(1) & \cdots & v^{\mathrm{T}}(N) \end{bmatrix}^{\mathrm{T}}$$
$$y_N = \begin{bmatrix} y^{\mathrm{T}}(0) & y^{\mathrm{T}}(1) & \cdots & y^{\mathrm{T}}(N) \end{bmatrix}^{\mathrm{T}}$$

The state transition matrix of system (1) can be given by

$$\begin{cases} \Phi(i+1, j) = A(i)\Phi(i, j) + A_\tau(i)\Phi(i-\tau, j), \\ \qquad\qquad\qquad\qquad i = 0, 1, \cdots, N-1, \ \ j \geq 0 \\ \Phi(m, n) = 0, \qquad\qquad m < n \\ \Phi(m, n) = I, \qquad\qquad m = n \end{cases} \tag{3}$$

Let

$$H_0 = \begin{bmatrix} C(0) \\ C(1)\Phi(1, 0) \\ C(2)\Phi(2, 0) \\ \vdots \\ C(N)\Phi(N, 0) \end{bmatrix}$$
$$H_{dN} = [h_{dN}(i, j)]_{(N+1)\times(N+1)}$$
$$H_{fN} = [h_{fN}(i, j)]_{(N+1)\times(N+1)}$$

with

$$h_{dN}(i, i) = 0, \quad h_{fN}(i, i) = D_f(i-1)$$

for $i = 1, 2, \cdots, N+1$, and

$$h_{\theta N}(i, j) = C(i-1)\Phi(i-1, j)B_\theta(j-1)$$

for $i > j = 2, 3, \cdots, N+1$; $\theta = d, \ f$ in sequence. We can rewrite (1) in the following form

$$y_N = H_{0N}x_0 + H_{dN}d_N + H_{fN}f_N + v_N \tag{4}$$

Under the assumption of x_0, $d(k)$, $v(k)$ being jointly normal distribution with (2) and $f(k) = 0$, the y_N is with normal distribution and has the following properties:

$$\mathcal{E}[y_N] = \mathcal{E}[H_{0N}x_0 + H_{dN}d_N + v_N] = 0 \tag{5}$$

$$R_{y_N} = \begin{bmatrix} H_{0N} & H_{dN} & I \end{bmatrix} \times \begin{bmatrix} P_0 & 0 & 0 \\ 0 & R_N & 0 \\ 0 & 0 & Q_N \end{bmatrix} \begin{bmatrix} H_0^{\mathrm{T}} \\ H_{dN}^{\mathrm{T}} \\ I \end{bmatrix} \tag{6}$$

where

$$R_N = \text{diag}\{R(0), \ R(1), \ \cdots, \ R(N)\}$$
$$Q_N = \text{diag}\{Q(0), \ Q(1), \ \cdots, \ Q(N)\}$$

Without loss of generality, it is assumed that R_{y_N} is invertible. Introduce

$$J_N = y_N^{\mathrm{T}} R_{y_N}^{-1} y_N \tag{7}$$

Recall that y_N is normally distributed and satisfies (5) and (6) if the fault is not taken into account. So, in the case of no fault, the J_N follows central χ^2 distribution with freedom degrees of $(N+1)m$. In the faulty case, however, the mean of y_N may not be zero and the distribution of y_N may be changed also. Thus the faulty case J_N does not have the central χ^2 distribution. Therefore, we can use J_N as an evaluation function and develop FD strategies based on statistical χ^2 testing of J_N.

Moreover, it should be pointed out that the online computation burden of such an evaluation function is heavy. Especially, it is not an easy task to implement online FD for LDTV systems with delayed state with the increasing of N. To overcome this problem, techniques of orthogonal projection and innovation analysis in [39] will be applied.

Given measurement observations $\{y(k)\}_{k=0}^N$, we now formulate the problem of FD as: 1) to find a projection of $x(k)$ on space spanned by $\{y(j)\}_{j=0}^k$ and calculate J_N online in terms of Riccati recursion; 2) to perform FD by using statistical χ^2 testing of J_N.

Remark 1. Different from the traditional FD for LDTV systems with time delay, it is novel to generate the evaluation function by directly using quadratic form (7) instead of residual generation as usual. There is no doubt that the online calculation of J_N is a heavy burden for time-delay systems with the increasing of N. So, it will be of significance to derive the recursive computation of J_N by using projection and innovation analysis.

2 Main results

To overcome the heavy computation problem of J_N in (7), orthogonal projection on linear space spanned by $\{y(i)\}_{i=0}^k$ will be first considered. Denote by $\hat{x}(j|k)$ the projection of $x(j)$ $(j = k-\tau, k-\tau+1, \cdots, k, k+1)$ onto the linear space spanned by $\mathcal{L}\{y(i)\}_{i=0}^k$. Let

$$\hat{x}(k+1) = \hat{x}(k+1|k)$$

Define innovations as

$$e(j) = x(j) - \hat{x}(j|k) \tag{8}$$
$$P(k) = \langle e(k), e(k) \rangle \tag{9}$$
$$P(j, i) = \langle e(j), e(i) \rangle, \ \text{for } i \neq j \tag{10}$$
$$\tilde{y}(k) = y(k) - C(k)\hat{x}(k) \tag{11}$$

$$R_{\tilde{y}}(k)=\langle\tilde{y}(k),\tilde{y}(k)\rangle \tag{12}$$

It is known from [39] that the innovation sequence $\{\tilde{y}(i)\}_{i=0}^{k}$ forms an orthogonal basis of linear space $\mathcal{L}\{y(i)\}_{i=0}^{k}$ and

$$\mathcal{L}\{\tilde{y}(i)\}_{i=0}^{k}=\mathcal{L}\{y(i)\}_{i=0}^{k}$$

Hence, the projection $\hat{x}(k+1)$ can be calculated by

$$\begin{aligned}\hat{x}(k+1)=&\sum_{i=0}^{k}\langle x(k+1),\tilde{y}(i)\rangle R_{\tilde{y}}^{-1}(i)\tilde{y}(i)=\\&\sum_{i=0}^{k-1}\langle x(k+1),\tilde{y}(i)\rangle R_{\tilde{y}}^{-1}(i)\tilde{y}(i)+\\&\langle x(k+1),\tilde{y}(k)\rangle R_{\tilde{y}}^{-1}(k)\tilde{y}(k)=\\&A(k)\hat{x}(k)+A_\tau(k)\hat{x}(k-\tau)+K_p(k)\tilde{y}(k)+\\&A_\tau(k)\sum_{i=k-\tau}^{k-1}K_\tau(i)\tilde{y}(i)\end{aligned} \tag{13}$$

$$\hat{x}(i)=0,\quad i\le 0 \tag{14}$$

where

$$\begin{aligned}K_p(k)=&\langle x(k+1),\tilde{y}(k)\rangle R_{\tilde{y}}^{-1}(k)=\\&(A(k)\Theta(k)+A_\tau(k)\Theta_\tau(k))R_{\tilde{y}}^{-1}(k)\\K_\tau(i)=&\langle x(k-\tau),\tilde{y}(i)\rangle R_{\tilde{y}}^{-1}(i)=\\&\Theta_\tau(k,i)R_{\tilde{y}}^{-1}(i)\\R_{\tilde{y}}(i)=&C(i)P(i)C^{\mathrm{T}}(i)+Q(i)\\\Theta(k)=&\langle x(k),\tilde{y}(k)\rangle=P(k)C^{\mathrm{T}}(k)\\\Theta_\tau(k)=&\langle x(k-\tau),\tilde{y}(k)\rangle=P(k-\tau,k)C^{\mathrm{T}}(k)\\\Theta_\tau(k,i)=&\langle x(k-\tau),\tilde{y}(i)\rangle=P(k-\tau,i)C^{\mathrm{T}}(i),\\&i=k-\tau,k-\tau+1,\cdots,k\end{aligned}$$

It follows from (1) and (8) $\sim$ (14) that

$$\left\{\begin{aligned}&e(k+1)=A(k)e(k)+A_\tau(k)e(k-\tau)+B_d(k)d(k)-\\&\qquad K_p(k)\tilde{y}(k)-A_\tau(k)\sum_{i=k-\tau}^{k-1}K_\tau(i)\tilde{y}(i)\\&\tilde{y}(i)=C(i)e(i)+v(i),\quad i=k-\tau,k-\tau+1,\cdots,k\end{aligned}\right. \tag{15}$$

Moreover, $P(k+1)$ can be given by

$$\begin{aligned}P(k+1)=&\langle e(k+1),e(k+1)\rangle=\\&A(k)P(k,k+1)+A_\tau(k)P(k-\tau,k+1)+\\&B_d(k)R(k)B_d^{\mathrm{T}}(k)-\\&K_p(k)C(k)P(k,k+1)-\\&A_\tau(k)\sum_{i=k-\tau}^{k-1}K_\tau(i)C(i)P(i,k+1)\end{aligned} \tag{16}$$

where

$$\begin{aligned}P(i,k+1)=&\langle e(i),e(k+1)\rangle=\\&P(i,k)A^{\mathrm{T}}(k)+P(i,k-\tau)A_\tau^{\mathrm{T}}(k)-\\&P(i,k)C^{\mathrm{T}}(k)K_p^{\mathrm{T}}(k)-\\&\sum_{j=k-\tau}^{k-1}P(i,j)C^{\mathrm{T}}(i)K_\tau^{\mathrm{T}}(i)A_\tau^{\mathrm{T}}(k)\end{aligned} \tag{17}$$

$$\begin{aligned}P(k,k+1)=&\langle e(k),e(k+1)\rangle=\\&P(k)A^{\mathrm{T}}(k)+P^{\mathrm{T}}(k-\tau,k)A_\tau^{\mathrm{T}}(k)-\\&\Theta(k)K_p^{\mathrm{T}}(k)\end{aligned} \tag{18}$$

$$\begin{aligned}P(k-\tau,k+1)=&\langle e(k-\tau),e(k+1)\rangle=\\&P(k-\tau,k)A^{\mathrm{T}}(k)+\\&P(k-\tau)A_\tau^{\mathrm{T}}(k)-\Theta_\tau(k)K_p^{\mathrm{T}}(k)-\\&\sum_{i=k-\tau}^{k-1}\Theta_\tau(k,i)K_\tau^{\mathrm{T}}(i)A_\tau^{\mathrm{T}}(k)\end{aligned} \tag{19}$$

Recall that

$$\begin{aligned}\hat{y}(k)=&\mathrm{Proj}\{y(k)|\mathcal{L}\{y(i)\}_{i=0}^{k-1}\}=\\&\mathrm{Proj}\{y(k)|\mathcal{L}\{\tilde{y}(i)\}_{i=0}^{k-1}\}=\\&\sum_{i=0}^{k-1}\langle y(k),\tilde{y}(i)\rangle R_{\tilde{y}}^{-1}(i)\tilde{y}(i)\\y(k)=&\hat{y}(k)+\tilde{y}(k)=\\&\begin{bmatrix}l_{k,1}&l_{k,2}&\cdots&l_{k,k-1}&I\end{bmatrix}\tilde{y}_k\end{aligned}$$

where

$$l_{k,i}=\langle y(k),\tilde{y}(i)\rangle R_{\tilde{y}}^{-1}(i),\quad k=1,2,\cdots,N;\ i\le k-1$$

We then have

$$y_N=L_N\tilde{y}_N \tag{20}$$

$$R_{y_N}=\langle y_N,y_N\rangle=L_N\langle\tilde{y}_N,\tilde{y}_N\rangle L_N^{\mathrm{T}}=L_NR_{\tilde{y}_N}L_N^{\mathrm{T}} \tag{21}$$

where L_N is given by

$$L_N=\begin{bmatrix}I&0&0&0\\l_{2,1}&I&0&0\\\vdots&\ddots&\ddots&\vdots\\l_{N+1,1}&\cdots&l_{N+1,N}&I\end{bmatrix} \tag{22}$$

with

$$l_{i,j}=\langle y(i),\tilde{y}(j)\rangle R_{\tilde{y}}^{-1}(j),\quad i=2,3,\cdots,N+1;\ j<i$$

and $R_{\tilde{y}_N}$ is block diagonal with

$$R_{\tilde{y}_N}=\mathrm{diag}\{R_{\tilde{y}}(0),R_{\tilde{y}}(1),\cdots,R_{\tilde{y}}(N)\} \tag{23}$$

Substituting (20) $\sim$ (23) to (7) yields

$$\begin{aligned}J_N=&\tilde{y}_N^{\mathrm{T}}L_N^{\mathrm{T}}R_{y_N}^{-1}L_N\tilde{y}_N=\tilde{y}_N^{\mathrm{T}}R_{\tilde{y}_N}^{-1}\tilde{y}_N=\\&\sum_{k=0}^{N}\tilde{y}^{\mathrm{T}}(k)R_{\tilde{y}}^{-1}(k)\tilde{y}(k)\end{aligned} \tag{24}$$

When a fault is taken into account, the J_N can be calculated by (24) with

$$\left\{\begin{aligned}&\hat{x}(k+1)=A(k)\hat{x}(k)+A_\tau(k)\hat{x}(k-\tau)+\\&\qquad A_\tau(k)\sum_{i=k-\tau}^{k-1}K_\tau(i)\tilde{y}(i)+K_p(k)\tilde{y}(k)\\&\tilde{y}(i)=y(i)-C(i)\hat{x}(i),\quad k-\tau\le i\le k\end{aligned}\right. \tag{25}$$

where $K_p(k)$, $K_\tau(i)$, $i=k-\tau,k-\tau+1,\cdots,k-1$ are calculated by

$$K_p(k) = (A(k)P(k)C^{\mathrm{T}}(k) + A_\tau(k)P(k-\tau,k)C^{\mathrm{T}}(k))R_{\tilde{y}}^{-1}(k) \quad (26)$$

$$K_\tau(i) = P(k-\tau,i)C^{\mathrm{T}}(i)R_{\tilde{y}}^{-1}(i) \quad (27)$$

$$R_{\tilde{y}}(k) = C(k)P(k)C^{\mathrm{T}}(k) + Q(k) \quad (28)$$

and $P(k)$ is given by (16) $\sim$ (19).

Based on the above analysis, the determination of evaluation function for FD can be summarized as the following theorem.

Theorem 1. Given observations $\{y(k)\}_{k=0}^N$, if there exists $P(k)$ given by (16) $\sim$ (19) such that $R_{\tilde{y}}(k) = C(k) \times P(k)C^{\mathrm{T}}(k) + Q(k) > 0$, then J_N can be calculated by

$$J_N = \sum_{k=0}^{N} \tilde{y}^{\mathrm{T}}(k)R_{\tilde{y}}^{-1}(k)\tilde{y}(k) \quad (29)$$

where $\tilde{y}(k)$ is generated by (25) with (16) $\sim$ (19) and (26) $\sim$ (28).

So far, we have derived a feasible evaluation function for FD. The remaining task is decision making, which consists of the choice of threshold and logic unit. Similar to the statistical testing of residuals in [3], the occurrence of a fault can be detected based on χ^2 testing.

For this purpose, we rewrite (25) in the following form

$$\begin{cases} e(k+1) = (A(k) - K_p(k)C(k))e(k) + A_\tau(k)e(k-\tau) - \\ \qquad A_\tau(k)\sum\limits_{i=k-\tau}^{k-1} K_\tau(i)C(i)e(i) + \\ \qquad B_d(k)d(k) + (B_f(k) - K_p(k)D_f(k))f(k) - \\ \qquad K_p(k)v(k) - A_\tau(k)\sum\limits_{i=k-\tau}^{k-1} K_\tau(i)v(i) \\ \tilde{y}(k) = C(k)e(k) + v(k) \end{cases}$$

It is easy to see that under the assumption of $d(k)$ and $v(k)$ being jointly normally distributed and satisfying (2), the elements of fault free case $\tilde{y}(k)$ are jointly normally distributed with

$$\mathcal{E}[\tilde{y}(k)] = 0$$
$$R_{\tilde{y}}(k) = C(k)P(k)C^{\mathrm{T}}(k) + Q(k)$$
$$\langle \tilde{y}(i), \tilde{y}(j) \rangle = 0, \quad i \neq j$$

Therefore, in the case of no fault, both the statistics $\tilde{y}(k)^{\mathrm{T}}R_{\tilde{y}}^{-1}(k)\tilde{y}(k)$ and J_N follow the central χ^2 distribution with degrees of freedom m and $(N+1)m$, respectively.

On the other hand, if the fault free case evaluation function follows the central χ^2 distribution with degree of freedom β, then a corresponding threshold with the false alarm rate ε can be chosen as $J_{\mathrm{th}} = \chi^2_{\beta,\varepsilon}$. Finally, the fault detection test can be performed as the following statistic χ^2 testing

$$\begin{cases} \text{if } J_e < \chi^2_{\beta,\varepsilon}, & \text{then no fault} \\ \text{if } J_e \geq \chi^2_{\beta,\varepsilon}, & \text{then fault alarm} \end{cases} \quad (30)$$

Similar to the case of statistical testing of residual in [3], the following three alternative strategies of FD are considered in this paper.

1) Single observation. For the given false alarm rate $\varepsilon > 0$, choose the evaluation function $J_e(k)$ and threshold $J_{\mathrm{th}}(k)$ as

$$\begin{cases} J_e(k) = \tilde{y}^{\mathrm{T}}(k)R_{\tilde{y}}^{-1}(k)\tilde{y}(k) \\ J_{\mathrm{th}} = \chi^2_{m,\varepsilon} \end{cases} \quad (31)$$

where $R_{\tilde{y}}(k)$ is given in (25).

2) Observation sequence. Choose the evaluation function J_{eN} and threshold J_{th} as

$$\begin{cases} J_{ek} = \tilde{y}_k^{\mathrm{T}} R_{\tilde{y}_k}^{-1} \tilde{y}_k \\ J_{\mathrm{th}}(k) = \chi^2_{(1+k)m,\varepsilon} \end{cases} \quad (32)$$

3) Window average of observations. Choose the evaluation function $J_{eN,a}$ and threshold $J_{\mathrm{th},a}$ as

$$\begin{cases} J_{ek,a} = \dfrac{1}{k+1}\tilde{y}_k^{\mathrm{T}} R_{\tilde{y}_k}^{-1} \tilde{y}_k \\ J_{\mathrm{th},a} = \chi^2_{m,\varepsilon} \end{cases} \quad (33)$$

Remark 2. When a residual signal $r(k)$ is chosen as $r(k) = R_{\tilde{y}}^{-1/2}(k)\tilde{y}(k)$, it is easy to have

$$J_N = \sum_{k=0}^{N} r^{\mathrm{T}}(k)r(k)$$

Moreover, the fault free case $r(k)$ is jointly normally distributed with

$$\mathcal{E}[r(k)] = 0, \quad \langle r(k), r(k) \rangle = I$$

In this case, one can further use the strategy in [3] to detect the occurrence of the fault, i.e. statistical testing of residual. Note that the main focus of this paper is the evaluation function J_N, but most of the existing results are concerned with the residual $r(k)$ or $\tilde{y}(k)$. From this viewpoint, we call the case of this paper the evaluation function based FD, while the conventional one as in [3] the residual based FD.

Remark 3. It is easy to see that (25) with $r(k) = R_{\tilde{y}}^{-1/2}(k)\tilde{y}(k)$ can be used as an observer-based FDF for the LDTV time-delay systems (1). Different from the most existing results for time-delay systems, not only the current innovation $\tilde{y}(k)$, but also the delayed innovations $\{\tilde{y}(i)\}_{i=k-\tau}^{k-1}$ are considered in (25).

3 A numerical example

Consider system (1) with the following parameters

$$A(k) = \begin{bmatrix} 1.8\mathrm{e}^{-k} & 0.2\sin(k) \\ 0 & 0.6 \end{bmatrix}$$

$$A_\tau(k) = \begin{bmatrix} 0.7 & 0.3\cos(k) \\ 0 & 0.9 \end{bmatrix}$$

$$B_f(k) = \begin{bmatrix} 0.9 \\ 1.8 \end{bmatrix}$$

$$B_d(k) = \begin{bmatrix} 0.3 \\ 0.7 \end{bmatrix}$$

$$C(k) = \begin{bmatrix} 1.4 & 2.8 \end{bmatrix}$$

$$D_f(k) = 0$$

Set $\tau = 2$, $P(0) = I$, and $P(0,i) = 0$, $i = 1, 2, 3$. Suppose that the process noise $d(k)$ and the measurement noise $v(k)$ follow normal distribution with zero mean and variance 1. The fault $f(k)$ is simulated as

$$f(k) = \begin{cases} -1, & k = 20 \sim 40 \\ 1, & k = 60 \sim 80 \\ 0, & \text{otherwise} \end{cases}$$

To detect the occurrence of a fault, the first step is to generate J_N online by using Theorem 1. The next step is the so called χ^2 statistical testing. An acceptable false alarm rate is supposed to be 0.1. The above discussed three testing strategies are considered as follows.

1) Single observation. The fault free case and faulty case evaluation functions are calculated by (31), i.e., the $J_0(k)$ and $J_e(k)$ shown in Fig. 1. Recall that $J_0(k)$ follows the central χ^2 distribution with degrees of freedom 1. The threshold is chosen as $J_{\rm th} = \chi^2_{1,0.1} = 2.7055$. The result in Fig. 1 has shown that both false alarms and missed fault alarm exist in this case.

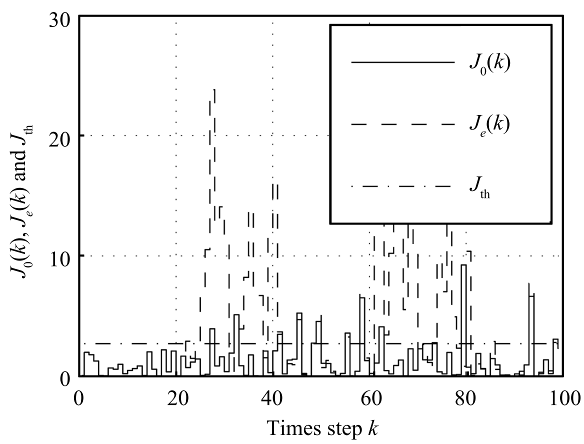

Fig. 1 The evaluation functions $J_0(k)$ and $J_e(k)$

2) Observation sequence. The fault free case and faulty case evaluation functions are calculated online by (32), i.e., the J_{0k} and J_{ek} shown in Fig. 2. Recall that J_{0k} follows the central χ^2 distribution with degree of freedom $k+1$. The threshold is chosen as $J_{\rm th}(k) = \chi^2_{k+1,0.1}$, i.e., the dash dot line in Fig. 2. It is easy to see that, in the faulty case, the fault alarm has 5 steps of delay and the fault alarm is not cleared away even if $f(k)$ is 0 for $41 \le k \le 59$ and $81 \le k \le 100$. Meanwhile, only a few false alarms appear in the fault free case.

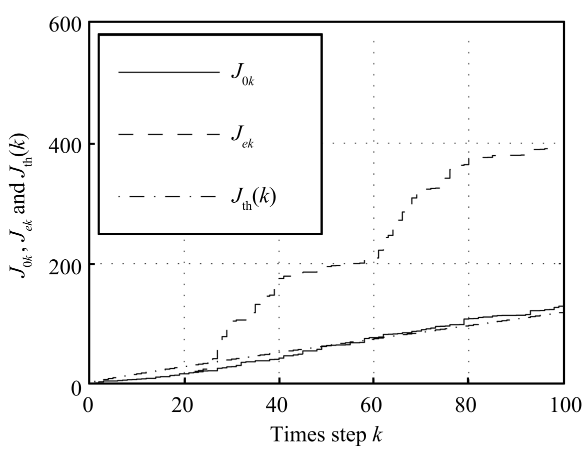

Fig. 2 The evaluation functions J_{0k} and J_{ek}

3) Window average of observations. The fault free case and faulty case evaluation function are calculated online by (33), i.e., the $J_{0k,a}$ and $J_{ek,a}$ shown in Fig. 3. Recall that $J_{0k,a}$ follows the central χ^2 distribution with degree of freedom 1. The threshold is chosen as $J_{{\rm th},a} = \chi^2_{1,0.1} = 2.7055$. It is seen from the result in Fig. 3 that in the faulty case, the fault alarm is delivered at $k = 29$ and is not cleared away even if $f(k)$ being 0 for $41 \le k \le 59$ and $81 \le k \le 100$. Meanwhile, there is no false alarm in the fault free case.

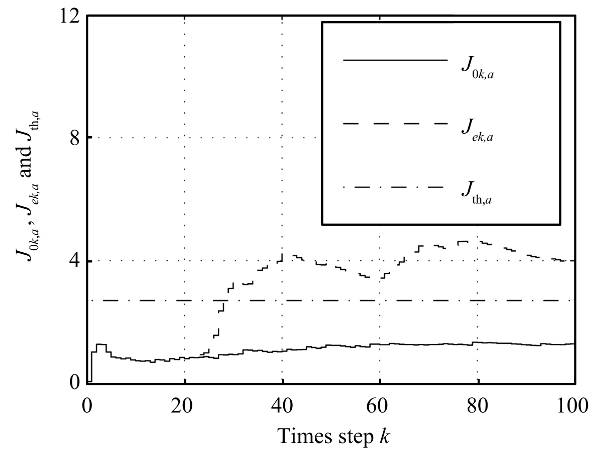

Fig. 3 The evaluation functions $J_{0k,a}$ and $J_{ek,a}$

Remark 4. It should be pointed out that the available χ^2 statistical testing strategies are not unique. In practical applications, we may choose one of them based on the trade-off between false alarm rate and missed alarm rate.

Remark 5. The residual mentioned in Remark 2 is also calculated and shown in Fig. 4.

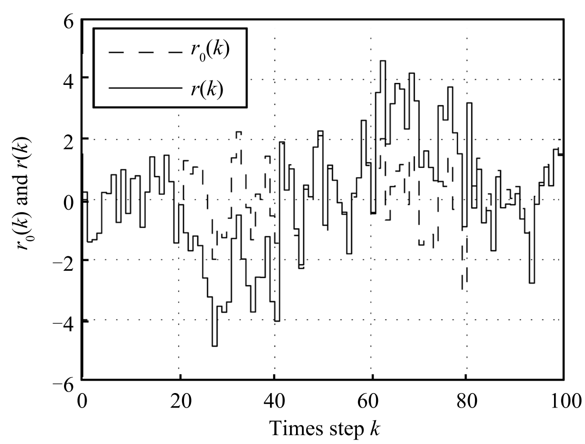

Fig. 4 The residuals $r_0(k)$ and $r(k)$

Since the fault free case residual follows zero mean normal distribution with variance 1, one can perform FD by using the statistical testing of residual $r(k)$ as in [3]. For simplicity, more details are not discussed here.

Remark 6. Using the single observation strategy, the occurrence of a fault can be detected in time, but false alarm is unavoidable. Meanwhile, the observation sequences and window average of observations strategies test the existence of fault over the considered time window, but the missed alarm is cleared at the start of fault happening. Moreover, the fault alarm can not stop in time even the fault disappears. So, for the purpose of practical application, a trade-off of false alarm and missed alarm rate is necessary.

4 Conclusion

In this paper, a statistical χ^2 testing based approach to FD has been proposed for a class of LDTV systems with delayed state and normally distributed noises. Under the assumption of process and measurement noises being with jointly normal distribution, a quadratic form of fault free case measurement output follows a central χ^2 distribution and, therefore, can be used as an evaluation function of FD. To reduce the heavy online computational burden, orthogonal projection and innovation analysis techniques are applied and, based on this, an available solution is obtained by recursively computing Riccati recursions. To detect the occurrence of a fault, three strategies of statistical χ^2 testing of evaluation function and decision making are discussed. Finally, a numerical example is given to illustrate the developed method.

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