A New Lyapunov Based Robust Control for Uncertain Mechanical Systems

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Abstract We design a new robust controller for uncertain mechanical systems. The inertia matrix′s singularity and upper bound property are first analyzed. It is shown that the inertia matrix may be positive semi-definite due to over-simplified model. Furthermore, the inertia matrix′s being uniformly bounded above is also limited. A robust controller is proposed to suppress the effect of uncertainty in mechanical systems with the assumption of uniform positive definiteness and upper bound of the inertia matrix. We theoretically prove that the robust control renders uniform boundedness and uniform ultimate boundedness. The size of the ultimate boundedness ball can be made arbitrarily small by the designer. Simulation results are presented and discussed.

Key words Inertia matrix, mechanical system, robust control, uncertainty

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To model a dynamic system more accurately, one must seek to capture more features of the system to render a better match between the model and the system. The captured features play an important role in the control design. Thus, validity of the captured features or properties of the system are foundations of modeling and control design.

For mechanical systems, it is essential to investigate the fundamental properties related to the control design. Then, we take advantage of the special structure as well as intrinsic properties of mechanical systems to design the controller. For example, the skew-symmetric matrix property[1] significantly reduces the work of control design for mechanical systems. In this paper, we explore more properties of mechanical systems which are useful in the control design.

A general robust control design is proposed in this paper to suppress the effect of uncertainty in mechanical systems. The controller is designed based on Lyapunov approach. For model uncertainty, here we assume it is possible to estimate the bound.

In early research, mechanical systems were treated as linear systems and PD or PID control was designed as feedback compensation. Since an actual mechanical system is a complex nonlinear dynamical system, therefore computed torque scheme was developed. However, there always exist unnoticeable and unknown aspects of the real system in the dynamic model which captures prominent features of the mechanical system. Diverse uncertainties include unknown parameters, unpredictable external disturbances, nonlinear friction, inertial cross coupling and so on. In general, uncertainties can be either stochastic or deterministic, and structured or unstructured. Researches on mechanical system control have always been very active, especially in handling uncertainties in the system. Adaptive control and robust control are the two major approaches among many exciting developments. They are both model-based. That is to say, a nominal system is selected at first and then the remaining portion of the system is lumped into the uncertainty. In the adaptive approach, adaption laws are constructed to explicitly learn the uncertain parameters. After employing robustness enhancement techniques, adaptive control can be applied to a wider range of uncertainties, but it is limited to systems with structured uncertainties.

Robust controller has a fixed structure that guarantees stability and performance for uncertain systems. It is capable of compensating for both structured and unstructured uncertainties and is simpler to implement. There are five major approaches currently available in robust control. These are H_∞[2−3], μ[4], Kharitonov[5−6], Lyapunov[7−8] and quantitative feedback theory (QFT)[9−10]. The first three are mainly for linear time-invariant systems. The QFT applies to nonlinear systems in practice but its theoretical basis remains to be justified. The Lyapunov approach is so far the only approach that has established theoretical basis and is applicable to non-autonomous nonlinear systems.

In this paper, the uncertainties in mechanical systems are assumed deterministic and Lyapunov based robust control algorithm is designed to render the mechanical system to follow a desired trajectory. We construct Lyapunov function through the inertia matrix to theoretically prove that this control renders uniform boundedness and uniform ultimate boundedness. In this robust controller, the maximum tolerance error between the actual and desired trajectories can be specified by the designer in advance. The control approach is based on assuming uniform positive definiteness and upper bound of the inertia matrix.

1 Fundamental properties of the mechanical system

1.1 Inertia matrix′s lower bound

We first review the example documented in McKerrow[11] where the inertia matrix

$$H(q)=\left[\begin{array}{cc} ml_2^2\cos^2\theta_2 & 0\\ 0 & ml_2^2\end{array}\right] \tag{1}$$

Thus $\det[H(q)]=0$ if $\theta_2=(2n+1)\frac{\pi}{2}$, $n=0,\pm1,\pm2,\cdots$. That is to say, the inertia matrix H is singular. When $\theta_2=(2n+1)\frac{\pi}{2}$, $n=0,\pm1,\pm2,\cdots$, the kinetic energy $\frac{1}{2}\dot{q}^{\mathrm T}H(q)\dot{q}=0$, $\forall\,\dot{\theta}_1$ which means the rotation does not increase kinetic energy. We point out that this is due to an over-simplified model and one may avoid this over-simplification by considering a more realistic model such as rigid bodies all of whose dimensions are non-negligible. However, in fact, much work on robot control employs particle mass model. One exam-

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ple is the two-link revolute joint manipulator in [12]. The inertia matrix′s non-singularity is not a criterion in the process of modeling. One usually simplifies the model based on his own judgements, so the inertia matrix′s singularity sometimes occurs.

For a slider-crank mechanism as seen in Fig. 1, based on one′s own judgements, the masses of crank and connecting rod are negligible. The slider′s mass is m. The slider can move in a plane and its position can be presented as

$$\boldsymbol{r} = (x \quad y) = (l_1 \cos\theta_1 + l_2 \cos\theta_2 \quad l_1 \sin\theta_1 + l_2 \sin\theta_2) \tag{2}$$

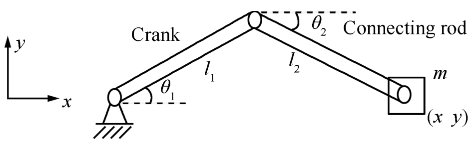

Fig. 1 Slider-crank mechanism

So, the Jacobian matrix and pseudo-inertia matrix are

$$J = \begin{bmatrix} \dfrac{\partial x}{\partial\theta_1} & \dfrac{\partial x}{\partial\theta_2} \\ \dfrac{\partial y}{\partial\theta_1} & \dfrac{\partial y}{\partial\theta_2} \end{bmatrix} = \begin{bmatrix} -l_1 \sin\theta_1 & -l_2 \sin\theta_2 \\ l_1 \cos\theta_1 & l_2 \cos\theta_2 \end{bmatrix}$$

$$M = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \tag{3}$$

The inertia matrix is

$$H = J^{\mathrm{T}} M J = \begin{bmatrix} -l_1 \sin\theta_1 & l_1 \cos\theta_1 \\ -l_2 \sin\theta_2 & l_2 \cos\theta_2 \end{bmatrix} \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \times \begin{bmatrix} -l_1 \sin\theta_1 & -l_2 \sin\theta_2 \\ l_1 \cos\theta_1 & l_2 \cos\theta_2 \end{bmatrix} = \begin{bmatrix} ml_1^2 & ml_1 l_2 \cos(\theta_1 - \theta_2) \\ ml_1 l_2 \cos(\theta_1 - \theta_2) & ml_2^2 \end{bmatrix} \tag{4}$$

When $\theta_1 - \theta_2 = n\pi$, $n = 0, \pm1, \pm2, \cdots$, $\det(H) = m^2 l_1^2 l_2^2 \sin^2(\theta_1 - \theta_2) = 0$. So, the inertia matrix is singular. In fact, if the masses of crank and connecting rod are considered, the inertia matrix becomes non-singular. However, modeling is a process based on one′s own judgements and the inertia matrix′s non-singularity is not a criterion in the process of modeling. So, the issue of inertia matrix′s singularity due to over-simplified modeling does exist and should be addressed.

1.2 Inertia matrix′s upper bound

To choose a legitimate Lyapunov function for control design, it has often been assumed by previous researchers that the inertia matrix is uniformly bounded above (i.e., $H \le \bar{\gamma} I$, where $\bar{\gamma} > 0$). However, it is limited to certain cases where the inertia matrix itself may be state-dependent. We take the two-degree-of-freedom manipulator in [12] as an example. The inertia matrix is

$$H = \begin{bmatrix} m_1 l_1^2 + I_1 + I_2 + m_2 d_2^2 & 0 \\ 0 & m_2 \end{bmatrix} \tag{5}$$

The generalized coordinate $q = [\theta_1 \ d_2]$. Obviously, H is not uniformly bounded above due to the d_2^2 term. So, we should explore more general property of the inertia matrix in the mechanical system.

Chen and Kuo[13] have proved and given their conclusion that for any inertia matrix $H(q)$ of any serial type mechanical manipulator, there exist constants γ_j, $j = 0, 1, 2$, with $\gamma_0 > 0$, $\gamma_{1,2} \ge 0$, such that

$$\|H(q)\| \le \gamma_0 + \gamma_1 \|q\| + \gamma_2 \|q\|^2, \quad \forall\, q \in \mathbf{R}^n \tag{6}$$

This upper bound property of the norm of the inertia matrix is generic. For any rigid serial type manipulators with revolute and prismatic joints, this property is applicable. We take this upper bound property as an assumption in mechanical systems. In the special case that all joints are revolute, the property is reduced to

$$\|H(q)\| \le \gamma_0, \quad \forall\, q \in \mathbf{R}^n \tag{7}$$

2 Mechanical systems with uncertainties

The general Lagrangian formulation of mechanical system dynamics in the form of matrix is

$$H(q(t))\ddot{q}(t) + V(q(t), \dot{q}(t)) + G(q(t)) + F(q(t), \dot{q}(t), t) = u(t) \tag{8}$$

In this equation of motion, $t \in \mathbf{R}$ is the time, $q(t) \in \mathbf{R}^n$ is the joint coordinate, $\dot{q}(t) \in \mathbf{R}^n$ is the joint velocity, $\ddot{q}(t) \in \mathbf{R}^n$ is the joint acceleration, $H(q(t))$ is the inertia matrix, $V(q(t), \dot{q}(t))$ is the Coriolis and centrifugal force, $G(q)$ is the gravitational force, $F(q(t), \dot{q}(t), t)$ represents Coulomb and viscous friction forces and external disturbances, and $u(t)$ is the generalized control force (we will omit arguments of functions where no confusions may arise from now on).

$V(q, \dot{q})$ can be factorized in such a way that

$$V(q, \dot{q}) = C(q, \dot{q})\dot{q} \tag{9}$$

with

$$\dot{H}(q) - 2C(q, \dot{q}), \quad \text{skew symmetric} \tag{10}$$

This factorization is always feasible and is independent of the parameter values of the mechanical system[1].

Assumption 1. We assume that in mechanical systems the inertia matrix $H(q)$ is uniformly positive definite (we emphasize that this is an assumption, not a fact), that is, there exists a scalar constant $\sigma > 0$ such that

$$H(q) \ge \sigma I, \quad \forall\, q \in \mathbf{R}^n \tag{11}$$

Assumption 2. For general mechanical systems, we assume that

$$\|H(q)\| \le \gamma_0 + \gamma_1 \|q\| + \gamma_2 \|q\|^2, \quad \forall\, q \in \mathbf{R}^n \tag{12}$$

In many practical situations, there may be modeling uncertainty and/or computational difficulty which prevents one from using the precise knowledge of M, C, G, and F. Uncertainty includes, for example, payload mass and friction force parameters. Here, we assume it is possible to estimate the bound of the model uncertainty.

3 The proposed robust controller

We wish the mechanical system to follow a desired trajectory $q^d(t)$, $t \in [t_0, t_1]$, with the desired velocity $\dot{q}^d(t)$. Assume $q^d(\cdot) : [t_0, \infty] \to \mathbf{R}^n$ is of class C^2 and $q^d(t)$, $\dot{q}^d(t)$ and $\ddot{q}^d(t)$ are uniformly bounded. Let

$$e(t) = q(t) - q^d(t) \tag{13}$$

and hence $\dot{e}(t) = \dot{q}(t) - \dot{q}^d(t)$, $\ddot{e}(t) = \ddot{q}(t) - \ddot{q}^d(t)$. Let

$$\underline{e}(t) = [e^{\mathrm{T}}(t) \ \ \dot{e}^{\mathrm{T}}(t)]^{\mathrm{T}} \tag{14}$$

The system (8) can be rewritten as

$$\begin{aligned}&H(e(t)+q^d(t))(\ddot{e}(t)+\ddot{q}^d(t))+\\&\quad C(e(t)+q^d(t),\dot{e}(t)+\dot{q}^d(t))(\dot{e}(t)+\dot{q}^d(t))+\\&\quad G(e+q^d)+F(e+q,\dot{e}+\dot{q}^d,t)=u(t)\end{aligned} \tag{15}$$

we first choose nominal matrices $\hat{H}$, $\hat{C}$, $\hat{G}$ and $\hat{F}$. Next, for a given $S = \mathrm{diag}\{[s_i]_{n\times n}\}$, $s_i > 0$, we choose (and hence we know) a scalar function $\rho : \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}_+$ such that

$$\rho(e, \dot{e}, t) \geq \|\phi(e, \dot{e}, t)\| \tag{16}$$

where

$$\begin{aligned}&\phi(e,\dot{e},t)=\\&\quad(\hat{H}(e+q^d(t))-H(e+q^d(t)))(\ddot{q}^d(t)-S\dot{e})+\\&\quad(\hat{C}(e+q^d,\dot{e}+\dot{q}^d)-C(e+q^d,\dot{e}+\dot{q}^d))(\dot{q}^d-Se)+\\&\quad\hat{G}(e+q^d)-G(e+q^d)+\\&\quad\hat{F}(e+q,\dot{e}+\dot{q}^d,t)-F(e+q,\dot{e}+\dot{q}^d,t)\end{aligned} \tag{17}$$

Here $\rho(e, \dot{e}, t)$ is based on the assumed bound of uncertainty. For a given $\epsilon > 0$ (often chosen to be "small") and $k_{p_i}, k_{v_i} > 0$, $i = 1, 2, \cdots, n$, the control torque is given by

$$\begin{aligned}u=\ &\hat{H}(\ddot{q}^d-S\dot{e})+\hat{C}(\dot{q}^d-Se)+\hat{G}+\hat{F}-\\&K_pe-K_v\dot{e}+p(e,\dot{e},t)\end{aligned} \tag{18}$$

where

$$p(e,\dot{e},t)=\begin{cases}-\dfrac{\mu(e,\dot{e},t)}{\|\mu(e,\dot{e},t)\|}\rho(e,\dot{e},t), & \text{if } \|\mu(e,\dot{e},t)\|>\epsilon\\ -\dfrac{\mu(e,\dot{e},t)}{\epsilon}\rho(e,\dot{e},t), & \text{if } \|\mu(e,\dot{e},t)\|\leq\epsilon\end{cases} \tag{19}$$

$$\mu(e, \dot{e}, t) = (\dot{e} + Se)\rho(e, \dot{e}, t) \tag{20}$$

$$K_p = \mathrm{diag}\{[k_{p_i}]_{n\times n}\} \tag{21}$$

$$K_v = \mathrm{diag}\{[k_{v_i}]_{n\times n}\} \tag{22}$$

4 Preliminaries of proposed controller

4.1 Positive definite

Lemma 1. Suppose $\phi : \mathbf{R}_+ \to \mathbf{R}_+$ is continuous, that $\phi(0) = 0$, ϕ is nondecreasing, and that $\phi(r) > 0$, $\forall\, r > 0$. Then there exists a function α of class K such that $\alpha(r) \leq \phi(r)$, $\forall\, r$. Moreover, if $\phi(r) \to \infty$ as $r \to \infty$, then α can be chosen to have the same property.

Proof. Pick a strictly increasing sequence q_i of positive numbers approaching infinity, and a strictly increasing sequence k_i of positive numbers approaching 1. Define

$$\alpha(r)=\begin{cases}\dfrac{r}{q_1}k_1\phi(r), & 0\leq r\leq q_1\\ k_i\phi(q_i)+\dfrac{r-q_i}{q_{i+1}-q_i}[k_{i+1}\phi(r)-k_i\phi(q_i)], & q_i<r\leq q_{i+1}\end{cases} \tag{23}$$

□

Definition 1. A function $V : \mathbf{R}_+ \times \mathbf{R}^n \to \mathbf{R}$ is said to be a locally positive definite function (LPDF) if

1) It is continuous;

2) $V(t, \mathbf{0}) = 0$, $\forall\, t \geq 0$;

3) There exist a constant $r > 0$ and a function α of class k such that

$$\alpha(\|\boldsymbol{x}\|) \leq V(t, \boldsymbol{x}), \quad \forall\, \boldsymbol{x} \in B_r \tag{24}$$

where B_r is the ball $B_r = \{x \in \mathbf{R}^n : \|x\| < r\}$.

Lemma 2. A continuous function $W : \mathbf{R}^n \to \mathbf{R}$ is an LPDF if and only if it satisfies the following two conditions: 1) $W(\mathbf{0}) = 0$; 2) there exists a constant $r > 0$ such that $W(\boldsymbol{x}) > 0$, $\forall\, \boldsymbol{x} \in B_r - 0$.

Proof. Suppose W is an LPDF in the sense of Definition 1, then clearly 1) and 2) above hold. To prove the converse, suppose 1) and 2) above are true, and define

$$\phi(p) = \inf_{p\leq\|\boldsymbol{x}\|\leq r} W(\boldsymbol{x}) \tag{25}$$

Then $\phi(0) = 0$, ϕ is continuous, and ϕ is nondecreasing because as p increases, the infimum is taken over a smaller region. Further, $\phi(p) > 0$ whenever $p > 0$; to see this, note that the annular region over which the infimum in (25) is taken is compact. Hence, if $\phi(p) = 0$ for some positive p, then there would exist a nonzero $\boldsymbol{x}$ such that $W(\boldsymbol{x}) = 0$, which contradicts 2). Now by Lemma 1, there exists an α of class K such that $\alpha(p) \leq \phi(p)$, $\forall\, p \in [0, r]$. By the definition of ϕ, it now follows that

$$\alpha(\|\boldsymbol{x}\|) \leq \phi(\|\boldsymbol{x}\|) \leq W(\|\boldsymbol{x}\|), \quad \forall\, \boldsymbol{x} \in B_r \tag{26}$$

Hence W is an LPDF in the sense of Definition 1. □

Lemma 3. A continuous function $V : \mathbf{R}_+ \times \mathbf{R}^n \to \mathbf{R}$ is an LPDF if and only if 1) $V(t, \mathbf{0}) = 0$, $\forall\, t$, and 2) there exists an LPDF $W : \mathbf{R}^n \to \mathbf{R}$ and a constant $r > 0$ such that

$$V(t, \boldsymbol{x}) \geq W(x), \quad \forall\, t \geq 0,\ \forall\, \boldsymbol{x} \in B_r \tag{27}$$

Proof. Suppose W is an LPDF and that (27) holds, then it is easy to verify that V is an LPDF in the sense of Definition 1. Conversely, suppose V is an LPDF in the sense of Definition 1, and let $\alpha(\cdot)$ be the function of class K such that (24) holds, then $W(\boldsymbol{x}) = \alpha(\|\boldsymbol{x}\|)$ is an LPDF such that (27) holds. □

4.2 Decrescent

Lemma 4. Suppose $\varphi : \mathbf{R}_+ \to \mathbf{R}_+$ is continuous, that $\varphi(0) = 0$, φ is nondecreasing. Then there exists a function β of class K such that $\beta(r) \geq \phi(r)$ $\forall\, r$. Moreover, if $\varphi(r) \to \infty$ as $r \to \infty$, then β can be chosen to have the same property.

Proof. Pick a strictly increasing sequence q_i of positive numbers approaching infinity, and a strictly decreasing sequence k_i of positive numbers approaching 1. Define

$$\beta(r)=\begin{cases}\dfrac{r}{q_1}k_1\varphi(r), & 0\leq r\leq q_1\\ k_i\varphi(q_i)+\dfrac{r-q_i}{q_{i+1}-q_i}[k_{i+1}\varphi(r)-k_i\varphi(q_i)], & q_i<r\leq q_{i+1}\end{cases} \tag{28}$$

□

Definition 2. A function $V : \mathbf{R}_+ \times \mathbf{R}^n \to \mathbf{R}$ is decrescent if there exist a constant $r > 0$ and a function β of class K such that

$$V(t,\boldsymbol{x}) \leq \beta(\|\boldsymbol{x}\|), \quad \forall\, t \geq 0,\ \forall\, \boldsymbol{x} \in B_r \tag{29}$$

Lemma 5. A continuous function $U : \mathbf{R}^n \to \mathbf{R}$ is decrescent if U is an LPDF.

Proof. If U is an LPDF in the sense of the Definition 1, then clearly $U(0) = 0$. Define

$$\varphi(p) = \sup_{0\leq\|\boldsymbol{x}\|\leq p\leq r} U(\boldsymbol{x}) \tag{30}$$

where "sup" represents supreme. Then $\varphi(0) = U(0) = 0$, φ is continuous and nondecreasing. By Lemma 4, there exists a β of class K such that $\varphi(p) \leq \beta(p)$, $\forall\, p \in [0, r]$. By the definition of φ, it now follows that

$$U(\|\boldsymbol{x}\|) \leq \varphi(\|\boldsymbol{x}\|) \leq \beta(\|\boldsymbol{x}\|), \quad \forall\, \boldsymbol{x} \in B_r \tag{31}$$

Hence U is decrescent in the sense of Definition 2. □

Lemma 6. A continuous function $V : \mathbf{R}_+ \times \mathbf{R}^n \to \mathbf{R}$ is decrescent if and only if there exists a decrescent function U: $\mathbf{R}^n \to \mathbf{R}$ and a constant $r > 0$ such that

$$V(t,\mathbf{x}) \leq U(\boldsymbol{x}), \quad \forall\, t \geq 0,\ \forall\, \boldsymbol{x} \in B_r \tag{32}$$

Proof. Suppose U is decrescent in the sense of Definition 2, then $U(\boldsymbol{x}) \leq \beta(\|\boldsymbol{x}\|)$. Also $V(t,\boldsymbol{x}) \leq U(\boldsymbol{x})$, then $V(t,\boldsymbol{x}) \leq U(\boldsymbol{x}) \leq \beta(\|\boldsymbol{x}\|)$. So, $V(t,\boldsymbol{x})$ is decrescent. To prove the converse, suppose V is decrescent in the sense of Definition 2, then there exists a $\beta(\cdot)$ of class K such that $V(t,\boldsymbol{x}) \geq \beta(\|\boldsymbol{x}\|)$. Pick $U(\boldsymbol{x}) = \beta(\|\boldsymbol{x}\|)$ which is decrescent. □

5 Theoretical proof of the controller

Theorem 1. Subject to Assumptions 1 and 2, the control (18) renders $\underline{e}(t)$ of the system (15) uniformly bounded and uniformly ultimately bounded. The size of the ultimate bounded ball can be made arbitrarily small by a suitable choice of ϵ.

Proof. Choose the function given by

$$\begin{aligned} V(t,\underline{e}) = \ & \frac{1}{2}(\dot{e}+Se)^{\mathrm{T}}M(e+q^d(t))(\dot{e}+Se) + \\ & \frac{1}{2}e^{\mathrm{T}}(K_p+SK_v)e \end{aligned} \tag{33}$$

1) "Positive definite" and "Decrescent"

To show that V is indeed a legitimate Lyapunov function candidate for any mechanical system, we shall prove that V is (globally) positive definite and decrescent. Based on (11),

$$\begin{aligned} V(t,\underline{e}) \geq \ & \frac{1}{2}\sigma\|\dot{e}+Se\|^2 + \frac{1}{2}e^{\mathrm{T}}(K_p+SK_v)e = \\ & \frac{1}{2}\sigma\sum_{i=1}^{n}(\dot{e}_i^2 + 2s_i\dot{e}_ie_i + s_i^2e_i^2) + \\ & \frac{1}{2}\sum_{i=1}^{n}(k_{p_i}+s_ik_{v_i})e_i^2 = \\ & \frac{1}{2}\sum_{i=1}^{n}[e_i \quad \dot{e}_i]\Psi_i\begin{bmatrix} e_i \\ \dot{e}_i \end{bmatrix} = W(\underline{e}) \end{aligned} \tag{34}$$

where

$$\Psi_i = \begin{bmatrix} \sigma s_i^2 + k_{p_i} + s_ik_{v_i} & \sigma s_i \\ \sigma s_i & \sigma \end{bmatrix} \tag{35}$$

and e_i and $\dot{e}_i$ are the ith components of e and $\dot{e}$, respectively. Since $\Psi_i > 0$, $\forall\, i$, we get

$$W(\underline{e}) \geq \frac{1}{2}\sum_{i=1}^{n}\lambda_{\min}(\Psi_i)(e_i^2+\dot{e}_i^2) \geq \frac{1}{2}\underline{\lambda}\,\|\underline{e}\|^2 = \alpha(\|\underline{e}\|) \tag{36}$$

where

$$\underline{\lambda} = \min_i \lambda_{\min}(\Psi_i), \quad i = 1,2,\cdots,n,\ \underline{\lambda} > 0 \tag{37}$$

Based on Lemma 3, we have shown that V is positive definite.

Based on (12),

$$\begin{aligned} V(t,\underline{e}) \leq \ & \frac{1}{2}(\gamma_0 + \gamma_1\|q\| + \gamma_2\|q\|^2)\|\dot{e}+Se\|^2 + \\ & \frac{1}{2}e^{\mathrm{T}}(K_p+SK_v)e = U(\underline{e}) \end{aligned} \tag{38}$$

Using (13), one has

$$\|q\| = \left\|e + q^d(t)\right\| \leq \|e\| + \max_t\left\|q^d(t)\right\| \tag{39}$$

$$\|q\|^2 \leq \|e\|^2 + 2\max_t\left\|q^d(t)\right\|\|e\| + \left(\max_t\left\|q^d(t)\right\|\right)^2 \tag{40}$$

$$\begin{aligned} & \|\dot{e}+Se\|^2 = \|\dot{e}+Se\|^{\mathrm{T}}\|\dot{e}+Se\| = \\ & \qquad \begin{bmatrix} e^{\mathrm{T}} & \dot{e}^{\mathrm{T}} \end{bmatrix}\begin{bmatrix} S^2 & S \\ S & I \end{bmatrix}\begin{bmatrix} e \\ \dot{e} \end{bmatrix} \leq \\ & \qquad \lambda_{\max}\begin{bmatrix} S^2 & S \\ S & I \end{bmatrix}\|\underline{e}\|^2 = \overline{S}\,\|\underline{e}\|^2 \end{aligned} \tag{41}$$

$$e^{\mathrm{T}}(K_p+SK_v)e \leq \lambda_{\min}(K_p+SK_v)\|e\|^2 \tag{42}$$

Note that $\overline{S} > 0$. Therefore there are constants $\overline{\lambda}_0 > 0$, $\overline{\lambda}_{1,2} \geq 0$ given by

$$\begin{aligned} \overline{\lambda}_0 = \ & \frac{1}{2}\lambda_{\max}(K_p+SK_v) + \\ & \frac{1}{2}\overline{S}\left[\gamma_0 + \gamma_1\max_t\left\|q^d(t)\right\| + \gamma_2\left(\max_t\left\|q^d(t)\right\|\right)^2\right] \end{aligned} \tag{43}$$

$$\overline{\lambda}_1 = \frac{1}{2}\overline{S}\left(\gamma_1 + 2\gamma_2\max_t\left\|q^d(t)\right\|\right) \tag{44}$$

$$\overline{\lambda}_2 = \frac{1}{2}\overline{S}\gamma_2 \tag{45}$$

such that

$$U(\underline{e}) \leq \overline{\lambda}_0\|\underline{e}\|^2 + \overline{\lambda}_1\|\underline{e}\|^3 + \overline{\lambda}_2\|\underline{e}\|^4 \leq \beta(\|\underline{e}\|) \tag{46}$$

Based on Lemma 6, V is decrescent for all $\underline{e} \in \mathbf{R}^{2n}$. Thus, we have shown that V is a legitimate Lyapunov function candidate.

2) To prove stability

Taking the time derivative of V along the trajectory of (15) yields

$$\begin{aligned} \dot{V} = \ & (\dot{e}+Se)^{\mathrm{T}}(u - H\ddot{q}^d - C\dot{e} - C\dot{q}^d - G - F + HS\dot{e}) + \\ & \frac{1}{2}(\dot{e}+Se)^{\mathrm{T}}\dot{H}(\dot{e}+Se) + e^{\mathrm{T}}(K_p+SK_v)\dot{e} = \\ & (\dot{e}+Se)^{\mathrm{T}}(u - H(\ddot{q}^d - S\dot{e}) - C(\dot{q}^d - Se) - G - F) - \\ & (\dot{e}+Se)^{\mathrm{T}}C(\dot{e}+Se) + \frac{1}{2}(\dot{e}+Se)^{\mathrm{T}}\dot{H}(\dot{e}+Se) + \\ & e^{\mathrm{T}}(K_p+SK_v)\dot{e} \end{aligned} \tag{47}$$

By (10) and using (18),

$$\begin{aligned}\dot{V} &= (\dot{e}+Se)^{\mathrm{T}}[(\hat{H}-H)(\ddot{q}^d-S\dot{e})+(\hat{C}-C)(\dot{q}^d-Se)+\\ &\quad \hat{G}-G+\hat{F}-F+p]-\dot{e}^{\mathrm{T}}K_v\dot{e}-e^{\mathrm{T}}SK_pe=\\ &\quad (\dot{e}+Se)^{\mathrm{T}}\phi+(\dot{e}+Se)^{\mathrm{T}}p-\dot{e}^{\mathrm{T}}K_v\dot{e}-e^{\mathrm{T}}SK_pe\end{aligned} \tag{48}$$

If $\|\mu\|>\epsilon$,

$$(\dot{e}+Se)^{\mathrm{T}}p=\frac{(\dot{e}+Se)^{\mathrm{T}}(\dot{e}+Se)\rho}{\|\mu\|}\rho=-\|\mu\| \tag{49}$$

If $\|\mu\|\le\epsilon$,

$$\begin{aligned}(\dot{e}+Se)^{\mathrm{T}}p&=\frac{(\dot{e}+Se)^{\mathrm{T}}(\dot{e}+Se)\rho}{\epsilon}\rho=\\ &-\frac{\|\dot{e}+Se\|^2\rho^2}{\epsilon}=-\frac{\|\mu\|^2}{\epsilon}\end{aligned} \tag{50}$$

Hence, by (16), (49), and (50),

$$(\dot{e}+Se)^{\mathrm{T}}\phi+(\dot{e}+Se)^{\mathrm{T}}p\le\|\mu\|+(\dot{e}+Se)^{\mathrm{T}}p\le\frac{\epsilon}{4} \tag{51}$$

Finally, upon using (51) in (48), we conclude that

$$\dot{V}\le-\underline{\lambda}_1\|\underline{e}(t)\|^2+\frac{\epsilon}{4} \tag{52}$$

for all $(\underline{e}(t),t)\in\mathbf{R}^{2n}\times\mathbf{R}$, where

$$\underline{\lambda}_1=\min\{\lambda_{\min}(K_v),\lambda_{\min}(SK_p)\} \tag{53}$$

□

The uniform boundedness performance is as follows[14]. That is, given any $r>0$ with $\|\underline{e}(t_0)\|\le r$, where t_0 is the initial time, there is a function $d(r)$ given by

$$d(r)=\begin{cases} r\left[\dfrac{2(\overline{\lambda}_0+\overline{\lambda}_1r+\overline{\lambda}_2r^2)}{\underline{\lambda}}\right]^{\frac{1}{2}}, & \text{if } r>\mathbf{R}\\ R\left[\dfrac{2(\overline{\lambda}_0+\overline{\lambda}_1R+\overline{\lambda}_2R^2)}{\underline{\lambda}}\right]^{\frac{1}{2}}, & \text{if } r\le\mathbf{R}\end{cases} \tag{54}$$

$$R=\left[\frac{\epsilon}{4\underline{\lambda}_1}\right]^{\frac{1}{2}} \tag{55}$$

such that $\|\underline{e}(t)\|\le d(r)$ for all $t\ge t_0$. The uniform ultimate boundedness performance also follows. That is, given any $\overline{d}$ with

$$\overline{d}>R\left[\frac{2(\overline{\lambda}_0+\overline{\lambda}_1R+\overline{\lambda}_2R^2)}{\underline{\lambda}}\right]^{\frac{1}{2}} \tag{56}$$

we have $\|\underline{e}(t)\|\le\overline{d}$, $\forall\, t\ge t_0+T(\overline{d},r)$, with

$$T(\overline{d},r)=\begin{cases}0, & \text{if } r\le\overline{R}\\ \dfrac{\overline{\lambda}_0r^2+\overline{\lambda}_1r^3+\overline{\lambda}_2r^4-\frac{1}{2}\underline{\lambda}\overline{R}^2}{\underline{\lambda}_1\overline{R}^2-\frac{\epsilon}{4}}, & \text{if } r\ge\overline{R}\end{cases} \tag{57}$$

$$\overline{R}=\gamma_2^{-1}(\frac{1}{2}\underline{\lambda}\overline{d}^2) \tag{58}$$

$$\gamma_2(\xi)=\overline{\lambda}_0\xi^2+\overline{\lambda}_1\xi^3+\overline{\lambda}_2\xi^4 \tag{59}$$

The ultimate bounded ball size $\overline{d}$ can be made arbitrarily small by a suitable choice of ϵ.

We note that in the proof $d(r)$ and $T(\overline{d},r)$ may be unknown due to the uncertain nature of $H(q)$ and therefore of $\overline{\lambda}_i$, $i=0,1,2$. However, one can easily calculate the upper bound of $\overline{\lambda}_i$ from the bound of λ_i in (54) if the bound of uncertainty is known. The upper bound of $\overline{\lambda}_i$ may be used in (54) and (57) to replace $\overline{\lambda}_i$ for a known $d(r)$ and $T(\overline{d},r)$.

Remark 1. The (positive) gain parameters s_i, k_{p_i}, and k_{v_i} are arbitrary. No restrictions are imposed. The designer has the discretion of choosing these parameters based on a number of practical factors such as the actual saturation limits.

Remark 2. In a sense, the matrices $\hat{H}$, $\hat{C}$, $\hat{G}$, and $\hat{F}$ define the torque for the nominal portion of the system. However, no restrictions on their choices are imposed. In the special case that there is no modeling uncertainty and one can afford sufficiently fast online computation, one naturally chooses $\hat{H}=H$, $\hat{C}=C$, $\hat{G}=G$, and $\hat{F}=F$ and therefore $\rho=0$, $\underline{e}(t)\to 0$ as $t\to\infty$.

6 Illustrative example

Consider a vehicle with an inverted pendulum hinged to the center as shown in Fig. 2. Assume that there is no friction between the vehicle and the ground. The vehicle′s mass is M (uncertain) and an external force F (the control) is imposed. The mass of the inverted pendulum is m (uncertain) and the length is l. An external torque τ (the control) is applied on the pendulum.

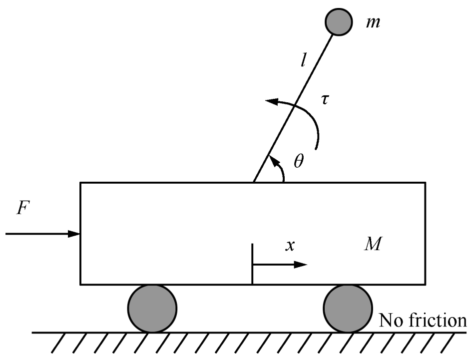

Fig. 2 Vehicle with an inverted pendulum

We choose two generalized coordinates $q=[q_1,q_2]^{\mathrm{T}}=[x,\theta]^{\mathrm{T}}$ to describe the mechanical system, where x denotes the displacement of the vehicle and θ denotes the rotatory angle of the pendulum. The two coordinates are independent of each other. The kinetic energy of the mechanical system is

$$T=\frac{1}{2}(M+m)\dot{x}^2+\frac{1}{2}ml^2\dot{\theta}^2-ml\dot{x}\dot{\theta}\sin\theta \tag{60}$$

The potential energy is

$$V=mgl\sin\theta \tag{61}$$

where g is gravitational acceleration. Then the Lagrange′s equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot{q}}\right)-\frac{\partial L}{\partial q}=u \tag{62}$$

can be written out where the Lagrangian $L=T-V$, u is the external control force. The equation of motion can be written in matrix form from using Lagrange′s equation as

$$H(q)\ddot{q}+C(q,\dot{q})\dot{q}+G(q)=u \tag{63}$$

where

$$q = \begin{bmatrix} x \\ \theta \end{bmatrix}, \quad \dot{q} = \begin{bmatrix} \dot{x} \\ \dot{\theta} \end{bmatrix}, \quad \ddot{q} = \begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix}, \quad u = \begin{bmatrix} F \\ \tau \end{bmatrix}$$

$$H(q) = \begin{bmatrix} M+m & -ml\sin\theta \\ -ml\sin\theta & ml^2 \end{bmatrix}$$

$$C(q,\dot{q}) = \begin{bmatrix} 0 & -ml\dot{\theta}\cos\theta \\ 0 & 0 \end{bmatrix}$$

$$G(q) = \begin{bmatrix} 0 \\ mgl\cos\theta \end{bmatrix} \tag{64}$$

The desired trajectory $q^d(t)$, the desired velocity and acceleration $\dot{q}^d(t)$, $\ddot{q}^d(t)$ are given by

$$q^d(t) = \begin{bmatrix} x^d \\ \theta^d \end{bmatrix} = \begin{bmatrix} \sin t \\ 1.5 - \cos t \end{bmatrix}$$

$$\dot{q}^d(t) = \begin{bmatrix} \dot{x}^d \\ \dot{\theta}^d \end{bmatrix} = \begin{bmatrix} \cos t \\ \sin t \end{bmatrix}$$

$$\ddot{q}^d(t) = \begin{bmatrix} \ddot{x}^d \\ \ddot{\theta}^d \end{bmatrix} = \begin{bmatrix} -\sin t \\ \cos t \end{bmatrix} \tag{65}$$

By using $q = e + q^d$, $\dot{q} = \dot{e} + \dot{q}^d$, $\ddot{q} = \ddot{e} + \ddot{q}^d$, (63) can be rewritten as

$$H(e+q^d)\ddot{e} + H(e+q^d)\ddot{q}^d + C(e+q^d, \dot{e}+\dot{q}^d)(\dot{e}+\dot{q}^d) + G(e+q^d) = u \tag{66}$$

where

$$H(e+q^d) = \begin{bmatrix} M+m & -ml\sin(e_2+1.5-\cos t) \\ -ml\sin(e_2+1.5-\cos t) & ml^2 \end{bmatrix}$$

$$C(e+q^d, \dot{e}+\dot{q}^d) = \begin{bmatrix} 0 & -ml(\dot{e}_2+\sin t)\cos(e_2+1.5-\cos t) \\ 0 & 0 \end{bmatrix}$$

$$G(e+q^d) = \begin{bmatrix} 0 \\ mgl\cos(e_2+1.5-\cos t) \end{bmatrix} \tag{67}$$

So, nominal matrices are given by

$$\hat{H} = \begin{bmatrix} \hat{M}+\hat{m} & -\hat{m}l\sin(e_2+1.5-\cos t) \\ -\hat{m}l\sin(e_2+1.5-\cos t) & \hat{m}l^2 \end{bmatrix}$$

$$\hat{C} = \begin{bmatrix} 0 & -\hat{m}l(\dot{e}_2+\sin t)\cos(e_2+1.5-\cos t) \\ 0 & 0 \end{bmatrix}$$

$$\hat{G} = \begin{bmatrix} 0 \\ \hat{m}gl\cos(e_2+1.5-\cos t) \end{bmatrix} \tag{68}$$

We choose S to be a 2×2 identity matrix. Therefore, we can get

$$\phi = (\hat{H}-H)(\ddot{q}^d - S\dot{e}) + (\hat{C}-C)(\dot{q}^d - Se) + \hat{G} - G =$$
$$\begin{bmatrix} \hat{M}-M+\hat{m}-m & -(\hat{m}-m)l\sin(e_2+1.5-\cos t) \\ -(\hat{m}-m)l\sin(e_2+1.5-\cos t) & (\hat{m}-m)l^2 \end{bmatrix} \times$$
$$\begin{bmatrix} -\sin t - \dot{e}_1 \\ \cos t - \dot{e}_2 \end{bmatrix} +$$
$$\begin{bmatrix} 0 & -(\hat{m}-m)l(\dot{e}_2+\sin t)\cos(e_2+1.5-\cos t) \\ 0 & 0 \end{bmatrix} \times$$
$$\begin{bmatrix} \cos t - e_1 \\ \sin t - e_2 \end{bmatrix} + \begin{bmatrix} 0 \\ (\hat{m}-m)gl\cos(e_2+1.5-\cos t) \end{bmatrix}$$

Based on ϕ, we choose the scalar function as

$$\rho = \left(\left\|\hat{M}-M\right\| + \|\hat{m}-m\|\right)\|\dot{e}_1 + \sin t\| +$$
$$\|\hat{m}-m\|\,\|l\sin(e_2+1.5-\cos t)(\dot{e}_2-\cos t)\| +$$
$$\|\hat{m}-m\|\,\|l(\dot{e}_2+\sin t)\cos(e_2+1.5-\cos t)(e_2-\sin t)\| +$$
$$\|(\hat{m}-m)\|\,\|l\sin(e_2+1.5-\cos t)(\dot{e}_1+\sin t)\| +$$
$$\|(\hat{m}-m)\|\,\left\|l^2(\dot{e}_2-\cos t)\right\| +$$
$$\|(\hat{m}-m)\|\,\|gl\cos(e_2+1.5-\cos t)\| \tag{69}$$

For simulation, we take

$$g=10,\ l=1,\ \epsilon=0.1,\ k_{p_1}=k_{p_2}=k_{v_1}=k_{v_2}=1$$
$$M=10,\ m=1,\ \hat{M}=10+\sin 10t,\ \hat{m}=1.1 \tag{70}$$

Two different classes of uncertainties, namely constant and high frequency, are chosen to test the control scheme. Furthermore, we compare the proposed robust control with the control scheme without p function (that is, computed-torque-like with PD control). We take the initial condition as

$$x(0)=1,\ \theta(0)=0.5,\ \dot{x}(0)=1,\ \dot{\theta}(0)=0.1$$
$$\ddot{x}(0)=1,\ \ddot{\theta}(0)=0.1 \tag{71}$$

Fig. 3 depicts the histories of x position errors of the proposed robust control (with p control) and computed-torque-like with PD control (without p control). We can see, with p control, the maximum overshoot is reduced from 1.64 to 1.5 and the settling time is much more reduced. Fig. 4 shows the histories of θ position errors of the two control schemes. With p control, the maximum overshoot is reduced and the improved control also has a smaller settling time and steady state error. In Fig. 5, p control has an increased maximum overshoot of the $\dot{x}$ velocity error, but the settling time is greatly reduced. Fig. 6 shows that, although p control has a bigger maximum overshoot of $\dot{\theta}$ velocity error, it has a greatly reduced steady state error. Fig. 7 shows that histories of $\ddot{x}$ acceleration error of the two control schemes are similar, while Fig. 8 illustrates that, with p control, the steady state error of $\ddot{\theta}$ acceleration error is much more reduced. Figs. 9 and 10 show histories of input force F and torque τ. We can see the histories are similar which means that the control costs are similar. According to the figures and analysis, we conclude that the proposed robust control gets a better performance compared to the control without p function.

7 Conclusions

The main contributions of this paper are twofold. First, the inertia matrix′s singularity and upper bound property are discussed in detail. The inertia matrix may be singular due to over-simplified modeling. The assertion of inertia matrix′s non-singularity remains to be an assumption. Second, a robust control scheme is proposed to deal with the uncertainty in mechanical systems. It is demonstrated that based on the non-singularity and upper bound assumptions of the inertia matrix, one can indeed utilize the inertia matrix to construct a legitimate Lyapunov function candidate for control design and stability analysis. The control does not need to have the uncertainty information of the dynamic system other than its upper bound.

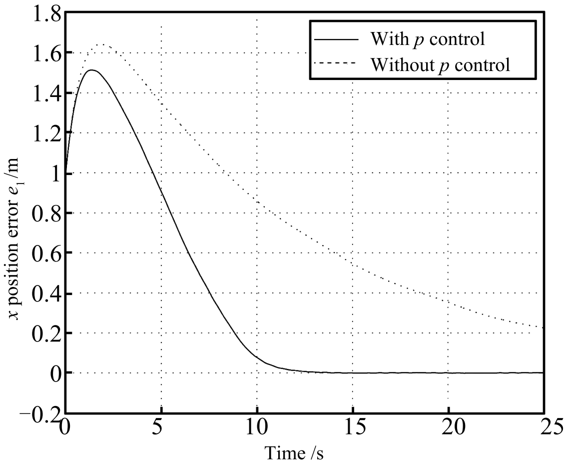

Fig. 3 Histories of x position errors e_1 of controls with p and without p

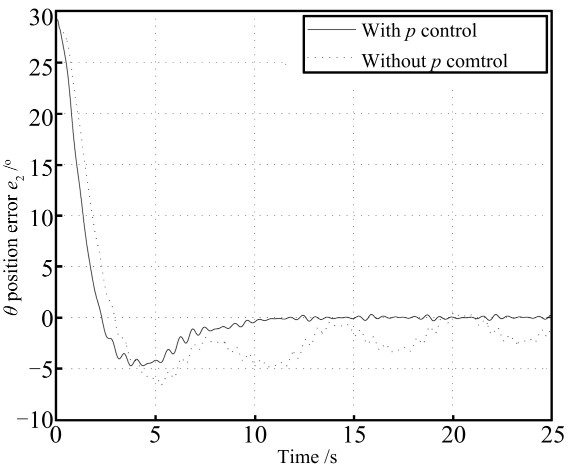

Fig. 4 Histories of θ position errors e_2 of controls with p and without p

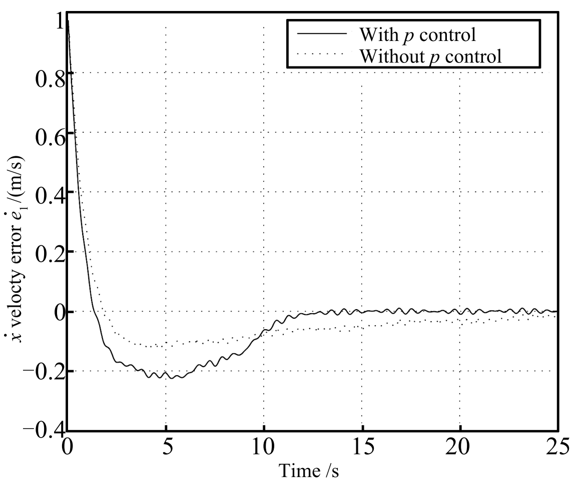

Fig. 5 Histories of $\dot{x}$ velocity errors $\dot{e}_1$ of controls with p and without p

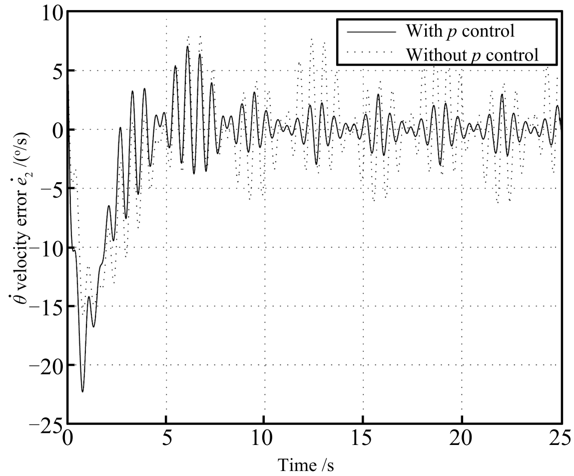

Fig. 6 Histories of $\dot{\theta}$ velocity errors $\dot{e}_2$ of controls with p and without p

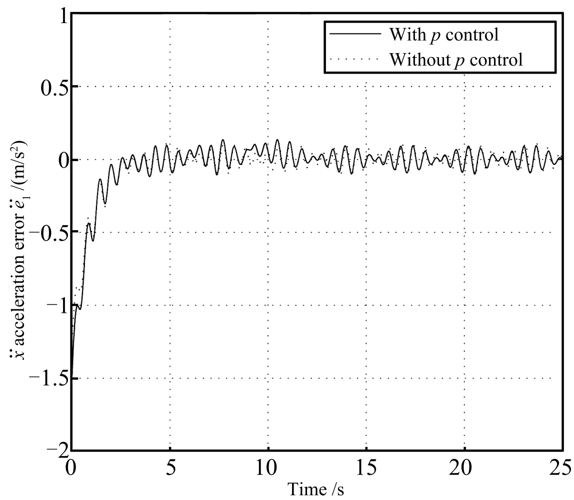

Fig. 7 Histories of $\ddot{x}$ acceleration errors $\ddot{e}_1$ of controls with p and without p

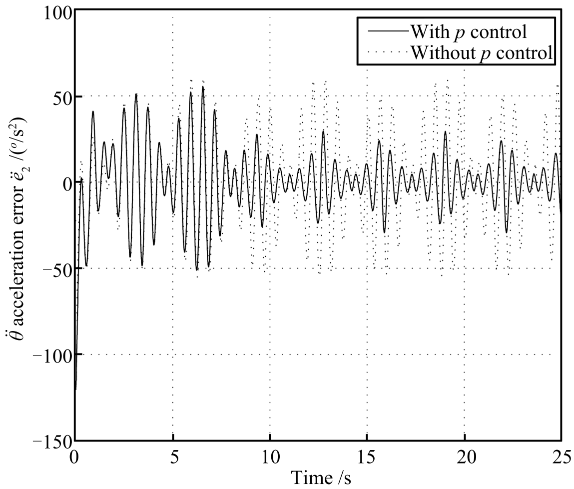

Fig. 8 Histories of $\ddot{\theta}$ acceleration errors $\ddot{e}_2$ of controls with p and without p

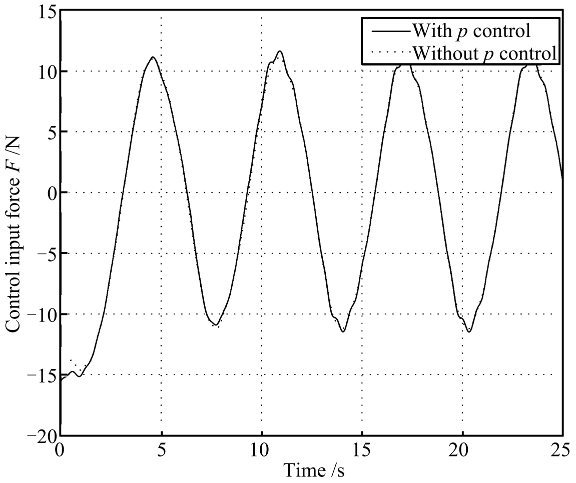

Fig. 9 History of control input force F

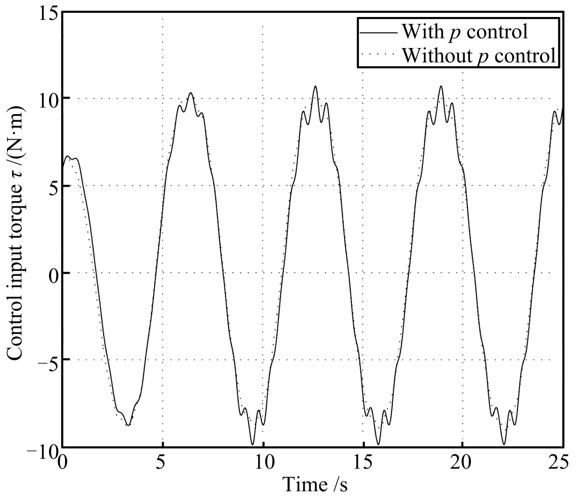

Fig. 10 History of control input torque τ

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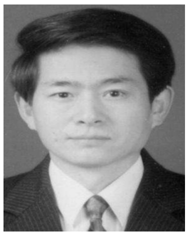

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