Quantized Nonlinear Control — A Survey

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Abstract Quantized control systems design is motivated by the convergence of controls and communications to address modern engineering applications involving the use of information technology. This paper presents an overview of recent developments on the control of linear and nonlinear systems when the control input is subject to quantization or the quantized states or outputs are used as feedback measurements. The co-existence of high-dimensionality, quantization, nonlinearity and uncertainty poses great challenges to quantized control of nonlinear systems and thus calls for new ideas and techniques. The field of quantized nonlinear control is still at its infancy. Preliminary results in our recent work based on input-to-state stability and cyclic-small-gain theorems are reviewed. The open problems in quantized nonlinear control are also outlined.

Key words Quantized control, nonlinear systems, input-to-state stability, small-gain theorem

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Information technology is commonly used in modern control engineering systems ranging from electric power grids to intelligent transportation systems and groups of unmanned (aerial, ground, and underwater) vehicles. For these control systems, signals are quantized before being transmitted via communication channels. The problem of controlling systems through quantized signals has recently raised a great interest within the controls community. Fig. 1 shows the block diagram of a typical quantized state-feedback control system.

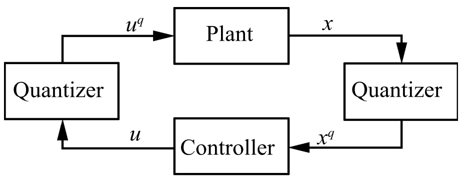

Fig. 1 The block diagram of a quantized control system (u is the control input computed by the controller, x is the state of the plant, u^q and x^q are quantized signals of u and x, respectively.)

A quantizer can be mathematically modeled as a discontinuous map from a continuous region to a discrete set of numbers. Two examples of widely adopted quantizers are shown in Sections 1 and 2. Quantization introduces strong nonlinear characteristics such as discontinuity and saturation to the system. It is for this reason that a direct application of the existing tools to quantized control design may not yield satisfactory results, particularly in the context of nonlinear systems.

1) Literature review

The study of quantized control can be traced back to the 1960′s[1−2]. Various results have been obtained for quantized control of linear systems since then. In [3], a discrete-event model of quantized control systems was developed. References [4−5] presented the results on minimum data rates for stabilizability of scalar systems, which have been subsequently extended for the stabilizability of autoregressive moving average (ARMA) systems[6] and linear systems described by state-space models[7−9]. If the number of quantization levels is not limited, then one may improve the control accuracy by increasing the quantization resolution (and equivalently, decreasing the quantization error) close to the origin. Moreover, it was shown in [10] that the logarithmic quantizer is the most efficient for stabilization with a quadratic Lyapunov function. In [11], the sector bound approach was used to study quantized control by formulating the quantization maps as sector-bounded uncertainties. Quadratic stability as well as H_2 and H_∞ performance criteria were also developed in [11]. References [12−13] studied the quadratic attractivity of quantized control systems with logarithmic quantizers and uniform quantizers. The robustness issues have also been taken into account in the study of quantized control. The quantized control problem with input and output quantization was considered in [14]. Reference [15] studied the robust stabilization problem for linear discrete-time systems via a limited capacity communication channel. In [16], it was proved that for scalar linear systems, the binary control strategy is the most robust with respect to varying data-rate constraint and asynchronism of sampling and control actuation. Quantized stabilization in the presence of additive disturbances was also been studied in [7, 9]. See also the survey paper [17] for the recent development of quantized control for linear systems.

All the papers mentioned above consider static quantization, that is, the quantization levels of the quantizers are fixed. For quantizers with finite numbers of quantization levels, improved control performance can be achieved by dynamically adjusting the quantization levels during the control process[18−22]. The dynamic quantization strategy developed in [20−21] is composed of two stages: zooming in and zooming out. Intuitively, if the system state is diverging from the reference, then the zooming-out stage is triggered and the range of the quantizer is increased to capture the system state; if the system state is converging to the reference, then the zooming-in stage is triggered and the quantization error is decreased for improved control accuracy. It should be noted that the adjustment of the quantization levels depends heavily on the divergence and convergence rates of the system state. Related problems were studied by [23−24] in the optimal control setting. References [7, 25−27] developed dynamic quantization strategies for robustness with respect to external disturbances. Specifically, the notion of input-to-state stability (ISS) invented by Sontag was used in [25, 27] to describe the influence of the external disturbances. The reader may consult [28] for the original development and [29] for a nice tutorial of ISS. Several related properties of ISS can also be found

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in [30−31]. Quantized control results for stochastic linear systems have also been developed; see e.g., [32−36] and the references therein.

Recently, there has been an increasing interest in quantized control of nonlinear systems[37−42]. Reference [37] considered nonlinear systems with quantized control inputs. In [41], the general idea of using (robust) control Lyapunov functions to design (robust) quantized controllers was proposed. In [42], the concept of the topological feedback entropy is used to analyze the data rate necessary to stabilize a nonlinear system. Reference [43] studied the conditions under which a logarithmic quantizer does not cancel the stabilizing effect of a continuous feedback control law for quantized control of dissipative systems. Set-valued maps were used in [43] to cover the sector bounded discontinuity caused by the logarithmic quantizer. The dynamic quantization strategy has also been extended to nonlinear systems. In [21, 40, 44], it was shown that if a system can be input-to-state stabilized with the quantization error as the input and the ISS gain satisfies a growth condition, then one can design a dynamic quantization strategy to realize global asymptotic stabilization. According [38], if the assumption of input-to-state stabilizability is relaxed by global asymptotic stabilizability, then semi-global asymptotic stabilization can be achieved. In [39], an n-bit encoded state feedback control law was recursively designed for a class of n-dimensional feedforward (or, upper-triangular) nonlinear systems. Quantized output-feedback control results can be found in [45−46].

It should be noted that the aforementioned issue of input-to-state stabilizability with respect to the quantization error is closely related to input-to-state stabilization with respect to sensor noise, which is far from being trivial for general nonlinear systems[47−50]. It is a well-known fact that even for low-dimensional nonlinear systems, small measurement errors may lead to instability[48, 51]. Unlike the actuator disturbance case[28, 52], the relationships between internal stabilizability and input-to-state stabilization in the presence of sensor noise still need further deeper investigation. Some special cases were studied for this issue of practical relevance and theoretic importance. For instance, the authors of [53] showed that for general nonlinear control systems under persistently acting disturbances, the existence of smooth Lyapunov functions is equivalent to the existence of (possibly discontinuous) feedback stabilizers which are robust with respect to small measurement errors and small additive external disturbances. Discontinuous controllers were also developed in [53] for nonlinear systems such that the closed-loop system is insensitive to small measurement errors. Reference [54] introduced a robust nonlinear control design approach based on backstepping methodology (see e.g., [55]) and flattened Lyapunov functions to deal with bounded measurement errors. But with the method in [54], the influence of the measurement errors grows with the order of the system, which is daunting for further application to quantized stabilization of high-order nonlinear systems. Additionally, [49] considered nonlinear systems composed of two subsystems, one is ISS and the other one is input-to-state stabilizable with respect to the measurement disturbance. In [49], the ISS of the closed-loop system was guaranteed by using the ISS small-gain theorem proposed in [56−57]. The gain assignment technique introduced in [50, 56] is employed such that the closed-loop system satisfies the small-gain condition. Related results can also be found in [58−61]. However, the results listed above are not directly applicable to the measurement feedback control problem for higher-dimensional nonlinear systems. On the other hand, another restrictive assumption in quantized stabilization[21, 40, 44] is that the ISS gain should also satisfy some growth-type condition.

With the recently developed ISS cyclic-small-gain theorem[62−63] as a tool, new methods for quantized stabilization of high-order nonlinear systems in the popular strict-feedback and output-feedback forms[55] were developed in [64−67]. Unlike the context of nonquantized nonlinear control, a direct application of the backstepping approach seems problematic because the virtual control laws based on quantized signals are not differentiable. Instead, the basic idea presented in our work[64−67] is to convert the quantized control problem into a network stability problem. More precisely, we first transform the control system into a new system of ISS subsystems via the use of appropriately designed virtual quantized control laws. Then, the cyclic-small-gain theorem is adopted to guarantee the ISS of the closed-loop systems. Also, the influence of quantization errors is captured explicitly by means of some ISS gains.

Another novel feature of quantized nonlinear control is that due to the discontinuity caused by quantization, the solutions of the closed-loop quantized systems cannot be defined as usual. Differential inclusion and the notion of Filippov solution[68] are used for stability analysis. Reference [69] proposed the extended Filippov solution for interconnected systems and extended the ISS small-gain theorem to discontinuous systems. ISS cyclic-small-gain results for large-scale dynamic networks of discontinuous subsystems can be directly developed by combining the results of [62−63, 69]. The discussion on the solutions of quantized systems in the sense of Carathéodory can be found in [43].

2) Organization of this paper

This survey paper aims to present an overview of the main ideas that form the basis of quantized control designs for nonlinear systems. We mainly review the ISS small-gain results for quantized nonlinear systems proposed in our recent work, in a unique objective to entice more efforts for the development of other design tools for quantized nonlinear control.

Section 1 is devoted to quantized nonlinear control with static quantization, while dynamic quantization is examined in Section 2. Section 2 also provides some discussions on the connections between the different types of quantization. In particular, the following issues are discussed.

1) How to transform a quantized control problem into an input-to-state stabilization problem;

2) How to use the ISS small-gain methods to realize input-to-state stabilization.

We also elaborate on the relationship of our small-gain approach with other existing design methods. Some future research directions are given in Section 3.

Most of the notations in this paper are standard. Some notations and definitions that are commonly used in the paper are given here. $\mathbf{R}^n$, $\mathbf{R}_+$ and $\mathbf{Z}_+$ represent the n-dimensional Euclidean space, the set of nonnegative real numbers and the set of nonnegative integers, respectively. $|x|$ represents the Euclidean norm of a vector $x \in \mathbf{R}^n$. A function $\alpha : \mathbf{R}_+ \to \mathbf{R}_+$ is said to be positive definite if $\alpha(0) = 0$ and $\alpha(s) > 0$ for $s > 0$. A continuous function $\alpha : \mathbf{R}_+ \to \mathbf{R}_+$ is said to be a class $\mathcal{K}$ function, denoted by $\alpha \in \mathcal{K}$, if it is strictly increasing and $\alpha(0) = 0$; it is said to be a class $\mathcal{K}_\infty$ function, denoted by $\alpha \in \mathcal{K}_\infty$, if it is a class $\mathcal{K}$ function and satisfies $\alpha(s) \to \infty$ as $s \to \infty$.

1 Static quantization

This section gives an overview of the quantized control results developed by [41, 43, 65, 70]. In these results, quantization error is considered as constrained uncertainty so that conventional robust control methods can be used. ISS has been recognized to be a powerful tool for robust control of nonlinear systems[29]. It is shown in this section that ISS offers a unified view in descrbing the basic idea of quantized nonlinear control.

To this end, we focus on the stabilization problem with quantized state feedback. Consider a nonlinear system

$$\dot{x} = f(x, u) \tag{1}$$

where $x \in \mathbf{R}^n$ is the state, $u \in \mathbf{R}^m$ is the control input. In the quantized control setting, the quantized signal $Q(x)$ with the quantizer $Q : \mathbf{R}^n \to \mathbf{R}^n$ generates the feedback measurement. The objective is to design a feedback control law in the form of

$$u = \varphi(Q(x)) \tag{2}$$

such that the closed-loop system is stabilized and the system state x converges to within some small neighborhood of the origin.

Define

$$\Delta(x) = Q(x) - x \tag{3}$$

as the quantization error. Then, the closed-loop system can be represented by

$$\dot{x} = f(x, \varphi(Q(x))) = f(x, \varphi(x + \Delta(x))) \tag{4}$$

With $\Delta(x)$ considered as the disturbance, the quantized control problem can thus be considered as a robust control problem.

Here, we consider quantizers satisfying the (truncated) sector bound property, which were studied in [41, 43, 65, 70]. Specifically, the quantizer Q can be represented by

$$Q(x) = [q_1(x_1), \cdots, q_n(x_n)]^{\mathrm{T}} \tag{5}$$

where each $q_i(x_i)$ satisfies

$$|q_i(x_i) - x_i| \leq b_i|x_i| + (1 - b_i)a_i \tag{6}$$

with constants $0 \leq b_i < 1$ and $a_i \geq 0$. Fig. 2 shows a truncated logarithmic quantizer, which satisfies property (6).

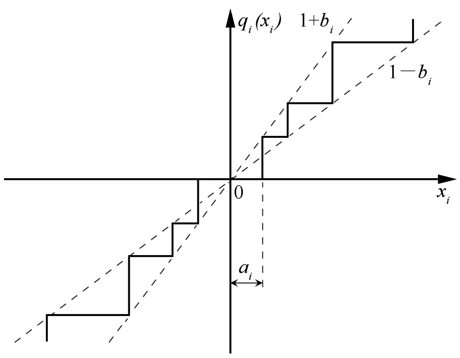

Fig. 2 A truncated logarithmic quantizer: $|q_i(x_i) - x_i| \leq b_i|x_i| + (1 - b_i)a_i$ for all $x_i \in \mathbf{R}$ with $a_i \geq 0$ and $0 \leq b_i < 1$

With property (6), one can find constants $a \geq 0$ and $0 \leq b < 1$ such that the perturbation $\Delta(x)$ caused by quantization satisfies

$$|\Delta(x)| \leq b|x| + (1 - b)a \tag{7}$$

1.1 Basic idea of the Lyapunov method

The quantized control objective is achievable if there exists a positive definite and radially unbounded $V : \mathbf{R}^n \to \mathbf{R}_+$ such that

$$\max_{|w| \leq b|x| + (1-b)a} \nabla V(x) f(x, \varphi(x + w)) \leq -\rho(|x|) + p \tag{8}$$

where $\rho \in \mathcal{K}_\infty$ and $p \geq 0$ is a constant. Here, V guarantees the practical stability of the closed-loop system in the presence of the uncertainty caused by quantization. Moreover, if $p = 0$, then the asymptotic stabilization is achieved. This general idea was originally developed in [41]. Note that a system with sufficiently small perturbation is locally stable at the origin if the corresponding perturbation-free system is asymptotically stable at the origin[71]. If the perturbation caused by quantization can be designed to be arbitrarily small, semi-global quantized stabilization can be obtained for asymptotically stabilizable systems[38]. Reference [43] yielded properties in the form of (8) for the closed-loop systems which are dissipative with w as the input. See, e.g., [72] for the basic knowledge of nonlinear systems.

1.2 ISS small-gain theorem as a tool

The discussions above are closely related to the notion of ISS[28], and can be performed by using ISS-Lyapunov functions. Below we show how ISS-Lyapunov function and the ISS small-gain theorem in [56] can be used in a systematic way to yield new solutions to quantized nonlinear control.

We give a condition for quantized stabilization of system (1) by using ISS-Lyapunov functions. Suppose that there exists a φ such that system $\dot{x} = f(x, \varphi(x + w))$ with w as the input is ISS with V as an ISS-Lyapunov function satisfying

$$\underline{\alpha}(|x|) \leq V(x) \leq \overline{\alpha}(|x|) \tag{9}$$

$$V(x) \geq \gamma(|w|) \Rightarrow \nabla V(x) f(x, \varphi(x + w)) \leq -\alpha(V(x)) \tag{10}$$

where $\underline{\alpha}, \overline{\alpha} \in \mathcal{K}_\infty$, $\gamma \in \mathcal{K}$, and α is a continuous and positive definite function.

Note that condition (7) implies that

$$|\Delta(x)| \leq \max\left\{(1 + \epsilon)b|x|, \left(1 + \frac{1}{\epsilon}\right)(1 - b)a\right\} \tag{11}$$

for any constant $\epsilon > 0$.

We consider the closed-loop quantized system as an interconnection of system $\dot{x} = f(x, \varphi(x + w))$ and the perturbation term $w = \Delta(x)$, as shown in Fig. 3.

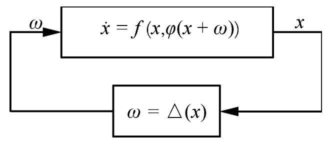

Fig. 3 The closed-loop quantized system

With the robust stability property of ISS[73] or the more general ISS small-gain theorem[56−57], if there exists an $\epsilon > 0$ such that

$$\underline{\alpha}^{-1}\circ\gamma((1+\epsilon)bs) < s \tag{12}$$

for all $s > 0$, then the closed-loop quantized system is practically stable at the origin. In the ideal case when $a = 0$, the closed-loop quantized system becomes asymptotically stable at the origin.

If we consider $\underline{\alpha}^{-1}\circ\gamma$ as the gain from w to x and $(1+\epsilon)b$ as the gain from w to x, then condition (12) means that the composition of the gains, or the loop gain, is less than the identity function and is called nonlinear small-gain condition. The system shown in Fig. 3 is composed of one dynamic system and one static system. Clearly, the nonlinear small-gain theorem in [56], originally developed for an interconnection of two dynamic systems which are ISS or more generally input-to-output stable (IOS), remains valid and applicable to this special scenario. Notice that the ISS small-gain theorem as a tool will also be shown powerful for quantized control of high-order nonlinear systems.

In the simple case shown in Fig. 3, the validity of condition (12) can be directly checked as follows. First, condition (12) means that $V(x) \geq \underline{\alpha}(|x|) > \gamma((1+\epsilon)b|x|)$. Then,

$$V(x) \geq \gamma\left(\left(1+\frac{1}{\epsilon}\right)(1-b)a\right) \Rightarrow V(x) \geq \gamma(|w|) \tag{13}$$

This together with property (10) implies that $V(x(t))$ ultimately converges to the region such that $V(x) \leq \gamma((1+1/\epsilon)(1-b)a)$. This means practical convergence. If $a = 0$, then asymptotic stabilization is realized.

1.3 Gain assignment technique for ISS small-gain design

In [38, 41, 43], it was assumed that an appropriate φ exists so that the closed-loop system (4) has specific stability properties. However, such assumption is restrictive for complex nonlinear systems. By extending the previous ISS small-gain design methods, it was shown in [65] that quantized input-to-state stabilization can be achieved for high-order nonlinear systems in the strict-feedback form. Due to the inherent robustness property of ISS, the design is also valid for systems with dynamic uncertainties and external disturbances. The essential feature of the small-gain method consists of transforming the closed-loop quantized system into a network of subsystems which are made by ISS by using a modified gain assignment technique and virtual quantized control laws. An important consequence of the modified gain assignment technique is that the ISS gains of the subsystems can be appropriately or somehow "arbitrarily" assigned. This way, for the transformed network of ISS subsystems, the cyclic-small-gain condition[62−63] holds so that the closed-loop quantized system is ISS. We refer the reader to [49−50, 56, 74] for the details on earlier versions of the gain assignment technique.

In this subsection, we employ the following first-order system to show the application of the gain assignment technique to quantized control:

$$\dot{x} = f(x,u) = \phi(x) + u \tag{14}$$

where $x \in \mathbf{R}$ is the state, $u \in \mathbf{R}$ is the control input, and $\phi : \mathbf{R} \to \mathbf{R}$ is an uncertain, locally Lipschitz function. Assume that there exists a known locally Lipschitz $\psi_\phi \in \mathcal{K}_\infty$ such that $|\phi(x)| \leq \psi_\phi(|x|)$ for all $x \in \mathbf{R}$. We show that the quantized stabilization of system (14) is solvable, if a $\varphi : \mathbf{R} \to \mathbf{R}$ can be found such that system (14) in closed-loop with the following control law

$$u = \varphi(x+w) \tag{15}$$

is ISS with w as the input and has an ISS-Lyapunov function satisfying conditions (9), (10) and (12).

The gain assignment technique developed in [49] can be readily used to solve the problem. The control law can be designed as $\varphi(r) = -\nu(|r|)r$ for $r \in \mathbf{R}$ with $\nu : \mathbf{R}_+ \to \mathbf{R}_+$ being a continuous, positive and nondecreasing function satisfying

$$(1-c)\nu((1-c)s)s \geq \frac{\ell}{2}s + \psi_\phi(s) \tag{16}$$

where $0 < c < 1$ and $\ell > 0$ are constants. Then, the closed-loop system composed of (14) and (15) is ISS with $V(x) = x^2/2$ as an ISS-Lyapunov function satisfying

$$\begin{aligned} &V(x) \geq \frac{|w|^2}{2c^2} \Rightarrow \\ &\qquad \nabla V(x)\left(\phi(x) + \varphi(x+w)\right) \leq -\ell V(x) \end{aligned} \tag{17}$$

Clearly, properties (9) and (10) are satisfied by $\underline{\alpha}(s) = \overline{\alpha}(s) = s^2/2$ and $\gamma(s) = s^2/2c^2$ for $s \in \mathbf{R}_+$. Condition (12) is satisfied if constants c and ϵ are chosen such that $c > (1+\epsilon)b$. With $0 \leq b < 1$, such c and ϵ exist.

1.4 High-order nonlinear systems

Over the last twenty years, nonlinear systems in the strict-feedback form[55] have been widely studied for adaptive and robust nonlinear control design, backed up by numerous practical examples. Among these novel tools is the ISS small-gain method that is proven useful in handling the dynamic uncertainties of systems[50, 56, 75].

A further step was taken in our recent work[65] to come up with a first solution to the quantized control problem for nonlinear uncertain systems in the strict-feedback form:

$$\dot{x}_i = x_{i+1} + \Delta_i(\bar{x}_i), \ \ 1 \leq i \leq n-1 \tag{18}$$

$$\dot{x}_n = u + \Delta_n(\bar{x}_n) \tag{19}$$

$$x_i^q = q_i(x_i), \ \ 1 \leq i \leq n \tag{20}$$

where $x = [x_1, \cdots, x_n]^{\mathrm{T}} \in \mathbf{R}^n$ is the state, $u \in \mathbf{R}$ is the control input, $\bar{x}_i = [x_1, \cdots, x_i]^{\mathrm{T}}$, x_i^q is the quantization of x_i, q_i's are state quantizers satisfying property (6), and the Δ_i's are unknown, locally Lipschitz functions. It is assumed that for each Δ_i, there exists a known $\psi_{\Delta_i} \in \mathcal{K}_\infty$ such that $|\Delta_i(\bar{x}_i)| \leq \psi_{\Delta_i}(|\bar{x}_i|)$ for all $\bar{x}_i \in \mathbf{R}^i$.

The control objective is to find, if possible, a quantized feedback controller in the form of

$$u = \varphi(x_1^q, \cdots, x_n^q) \tag{21}$$

such that the closed-loop solutions $x(t)$ are globally bounded and ultimately converge to within some small neighborhood of the origin.

If the accurate measurement of x is available for feedback, i.e., $q_i(x_i) = x_i$ for $i = 1, \cdots, n$, then the ISS small-gain design proposed in [50] can be readily used for stabilization. Specifically, one may recursively design a nonlinear controller as

$$x_1^* = \breve{\kappa}_1(x_1) \tag{22}$$

$$x_{i+1}^* = \breve{\kappa}_i(x_i - x_i^*), \quad i = 2, \cdots, n-1 \tag{23}$$

$$u = \breve{\kappa}_n(x_n - x_n^*) \tag{24}$$

where the $\breve{\kappa}_i'$s for $i = 1, \cdots, n$, viewed as "virtual control laws", are appropriately designed nonlinear functions, and u is the implementable control law.

To take into account the quantization, each x_i for $i = 1, \cdots, n$ in (22) ~ (24) is replaced with x_i^q and the quantized control law is designed in the form of

$$x_1^* = \kappa_1(x_1^q) \tag{25}$$
$$x_{i+1}^* = \kappa_i(x_i^q - x_i^*), \quad i = 2, \cdots, n-1 \tag{26}$$
$$u = \kappa_n(x_n^q - x_n^*) \tag{27}$$

where the κ_i's are not necessarily the same as the $\breve{\kappa}_i'$s in (22) ~ (24) and should be carefully designed to make the control law compatible with the sensor noise.

As shown below, the discontinuity of the quantization leads to a major difficulty in quantized control design. If $q_i(x_i) = x_i$ for $i = 1, \cdots, n$, then one may choose a coordinate transformation

$$e_1 = x_1 \tag{28}$$
$$e_i = x_i - \breve{\kappa}_{i-1}(e_{i-1}), \quad i = 2, \cdots, n \tag{29}$$

and design the feedback control law so that the closed-loop system with $e = [e_1, \cdots, e_n]^{\mathrm{T}}$ as the state is asymptotically stable[50]. The functions $\breve{\kappa}_i$ for $i = 1, \cdots, n-1$ are required to be continuously differentiable so that the new system state e is continuously differentiable. However, due to the discontinuity of quantization, one cannot directly replace the x_i in (28) and (29) with x_i^q for quantized control design.

The contribution of [65] lies in the novel set-valued map design to cover the influence of quantization. Specifically, in the presence of state quantization, the new state variables are defined as

$$e_1 = x_1 \tag{30}$$
$$e_i = \boldsymbol{d}(x_i, S_{i-1}(\bar{x}_{i-1})), \quad i = 2, \cdots, n \tag{31}$$

where

$$\boldsymbol{d}(z, \Omega) = z - \arg\min_{z' \in \Omega}\{|z - z'|\} \tag{32}$$

represents the directed distance from $z \in \mathbf{R}$ to compact set $\Omega \subset \mathbf{R}$, and for each $i = 1, \cdots, n-1$, $S_i : \mathbf{R}^i \to \mathbf{R}$ is an appropriately designed set-valued map depending on κ_i. Basically, the set-valued maps are employed to cover the influence of quantization. An example of the definition of e_2 is shown in Fig. 4. By choosing the boundaries of the set-valued maps to be continuously differentiable almost everywhere, the new state variable $e = [e_1, \cdots, e_n]^{\mathrm{T}}$ is continuously differentiable almost everywhere. It was shown in [65] that if $q_i(x_i) = x_i$ for $i = 1, \cdots, n$, then the coordinate transformation (30) and (31) is reduced to (28) and (29).

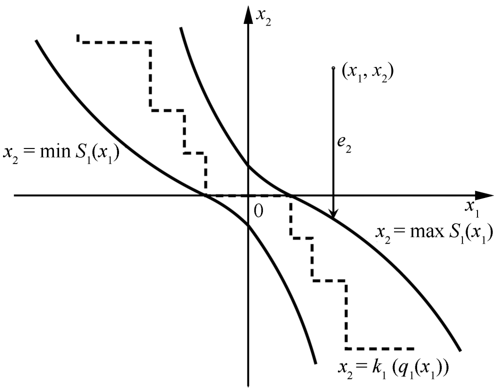

Fig. 4 The definitions of S_1 and e_2

With the set-valued map designs, the e_i-subsystems can be represented by differential inclusions

$$\dot{e}_i \in F_i(e_1, \cdots, e_{i+1}) \tag{33}$$

with $F_i : \mathbf{R}^{i+1} \to \mathbf{R}$, which can be designed to be ISS with $e_1, \cdots, e_{i-1}, e_{i+1}$ as the inputs by using a modified gain assignment technique. Specifially, $e_{n+1} = 0$. Thus, the closed-loop quantized system is transformed into a network of ISS subsystems. The interconnection structure of the network can be represented by a directed graph (digraph), as shown in Fig. 5. Then, the recently developed cyclic-small-gain theorem for large-scale dynamic networks[62] can be readily used to check the stability property of the closed-loop quantized system. See also [76−77] for a variety of small-gain results for very general nonlinear systems. Moreover, with the Lyapunov-based ISS cyclic-small-gain theorem [63], an ISS-Lyapunov function can be constructed for the closed-loop quantized system by using the ISS-Lyapunov functions of the e_i-subsystems to explicitly describe the convergence property.

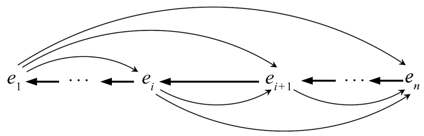

Fig. 5 The interconnection graph of the closed-loop quantized system

The truncated sector bound condition (7) assumed in this section means that the quantizers have infinite number of quantization levels. Due to finite-word length, a practical quantizer has a finite number of quantization levels, for which condition (7) may only hold in a specific range of x, say, Ω. If the parameters of the quantizers are fixed, then the above-described design could only guarantee local stabilization within Ω. If the upper-bound of the initial state of the system is known a priori and the quantizers are designable, then the semi-global stabilization problem is solvable. This idea was used in [38] and remains valid for the quantized stabilizer design based on the ISS small-gain theorem.

2 Dynamic quantization

As discussed in Section 1, if the quantizers have finite numbers of fixed quantization levels, then, generally speaking, global and semiglobal quantized stabilization can hardly be achieved via static quantization. Instead, we turn to use dynamic quantization that consists of dynamically adjusted quantization levels during the process of control for enlarged region of stability.

To highlight the relationship between the quantization error and the quantization range, in this section, we consider uniform quantizers with finite numbers of quantization levels. Fig. 6 shows an example. If the input r of the quantizer is within the quantization range $M\mu$ with M being a positive constant, then the quantization error is less than μ, i.e.,

$$|r| \le M\mu \Rightarrow |q(r, \mu) - r| \le \mu \tag{34}$$

Otherwise, the output of the quantizer is saturated. With μ considered as a variable of the quantizer, dynamic quantization is performed by adjusting μ in the control process. It should be noted that more general quantizers can be used for dynamic quantization with the methods reviewed in this section as long as property (34) holds; see, e.g., [78] for discussions.

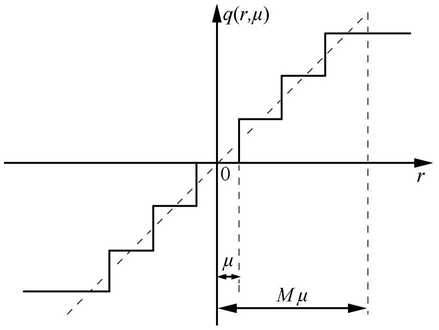

Fig. 6 A uniform quantizer q with a finite number of levels: μ represents the quantization error within the quantization range $M\mu$, i.e., $|r| \leq M\mu \Rightarrow |q(r,\mu) - r| \leq \mu$, with M being a positive integer

Dynamic quantization was originally developed in [20] for quantized control of linear systems by using Lyapunov arguments. With a Lyapunov-based ISS formulation, the basic idea is naturally extended to nonlinear systems; see, e.g., [21, 40, 44].

2.1 Basic idea

To clarify the basic idea of dynamic quantization, we consider the following closed-loop quantized system as an example:

$$\dot{x} = f(x, \varphi(q(x,\mu))) \tag{35}$$

where $x \in \mathbf{R}$ is the state, $f : \mathbf{R}^2 \to \mathbf{R}$ is a locally Lipschitz function, $\varphi : \mathbf{R} \to \mathbf{R}$ is the control law, $q : \mathbf{R} \times \mathbf{R}_+ \to \mathbf{R}$ is the quantizer as shown in Fig. 6 with parameter $M > 0$, and $\mu \in \mathbf{R}_+$ is the variable of the quantizer.

By defining quantization error $\Delta(x,\mu) = q(x,\mu) - x$, the closed-loop quantized system can be rewritten as

$$\dot{x} = f(x, \varphi(x + \Delta(x,\mu))) \tag{36}$$

Assume that system $\dot{x} = f(x, \varphi(x+w))$ is ISS with w as the input and admits an ISS-Lyapunov function $V : \mathbf{R} \to \mathbf{R}_+$ satisfying

$$\underline{\alpha}(|x|) \leq V(x) \leq \overline{\alpha}(|x|) \tag{37}$$

$$V(x) \geq \gamma(|w|) \Rightarrow \nabla V(x) f(x, \kappa(x+w)) \leq -\alpha(V(x)) \tag{38}$$

where $\underline{\alpha}$, $\overline{\alpha} \in \mathcal{K}_\infty$, $\gamma \in \mathcal{K}$ and α is a continuous and positive definite function. Here, w can be considered as the measurement error caused by quantization. Also assume that $\underline{\alpha}$, γ and M satisfy

$$\underline{\alpha}^{-1} \circ \gamma(\mu) \leq M\mu \tag{39}$$

for all $\mu \in \mathbf{R}_+$.

Consider the case of $\underline{\alpha}(M\mu) \geq V(x) \geq \gamma(\mu)$. Direct calculation yields:

$$V(x) \leq \underline{\alpha}(M\mu) \Rightarrow |x| \leq M\mu \Rightarrow |\Delta(x,\mu)| \leq \mu \tag{40}$$

which together with $V(x) \geq \gamma(\mu)$ implies $V(x) \geq \gamma(|\Delta(x,\mu)|)$. By using (38), we have

$$\begin{aligned} &\underline{\alpha}(M\mu) \geq V(x) \geq \gamma(\mu) \Rightarrow \\ &\qquad \nabla V(x) f(x, \kappa(x + \Delta(x,\mu))) \leq -\alpha(V(x)) \end{aligned} \tag{41}$$

Clearly, $\Omega_L(\mu) = \{x : V(x) \leq \underline{\alpha}(M\mu)\}$ and $\Omega_S(\mu) = \{x : V(x) \leq \gamma(\mu)\}$ are nested invariant sets of the closed-loop quantized system. The existence of such invariant sets is recognized as a sufficient condition for dynamic quantization[78]. Here, it can also be observed that if μ is fixed, then quantized stabilization can only be guaranteed within $\Omega_L(\mu)$.

Suppose that an upper bound of $V(x(0))$ is known a priori. By choosing $\mu(0)$ such that $\underline{\alpha}(M\mu(0)) \geq V(x(0))$ and reducing μ on the timeline slowly, asymptotic stabilization can be achieved. Fig. 7 shows the case where μ is updated on discrete time instants $\{t_k\}_{k \in \mathbf{Z}_+}$ satisfying $t_{k+1} - t_k = \delta_t$ with constant $\delta_t > 0$.

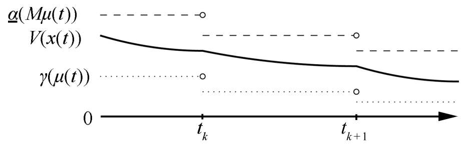

Fig. 7 Basic idea of dynamic quantization

The nested invariant sets defined by (41) play a central role in dynamically quantized control of nonlinear systems. The process of reducing μ is usually known as the zooming-in stage of dynamic quantization. Variable μ is also called zooming variable.

In the case where the upper bound of $V(x(0))$ is unknown, (semi)global stabilization can be achieved for forward complete systems by employing a zooming-out stage, i.e., increasing μ and thus $\underline{\alpha}(M\mu)$ being fast enough such that $\underline{\alpha}(M\mu(t^*)) \geq V(x(t^*))$ at some finite time t^*. Very detailed discussions of this idea can be found in [78]. Dynamic quantization for more general nonlinear systems can also be implemented based on this idea.

Note that condition (39) is equivalent to

$$\underline{\alpha}^{-1} \circ \gamma\left(\frac{1}{M}s\right) \leq s \tag{42}$$

for all $s \in \mathbf{R}_+$. By comparing (12) and (42), one may recognize the similarity between the conditions for static quantization and dynamic quantization.

For the first-order system (14), by using the gain assignment technique reviewed in Subsection 1.3, we can design a control law in the form of (15) such that with w as the input, the closed-loop system $\dot{x} = f(x, \varphi(x+w))$ is ISS with $V(x) = x^2/2$ as an ISS-Lyapunov function satisfying (37) and (38) with $\underline{\alpha}(s) = \overline{\alpha}(s) = s^2/2$ and $\gamma(s) = s^2/2c^2$ for $s \in \mathbf{R}_+$. The constant c can be arbitrarily chosen such that $0 < c < 1$. Condition (42) is satisfied if $cM > 1$. This is practically realizable as $M \geq 3$ holds for any uniform quantizer with no less than three levels[66].

Moreover, as shown in the next subsection, the ISS small-gain methods are also quite useful in solving the new problems with dynamic quantization for high-order nonlinear systems.

2.2 Quantized output-feedback control

To highlight the difficulty caused by the high-dimensionality of the system, this subsection discusses quantized output-feedback control with one quantizer. Consider the following nonlinear system in the output-feedback form with quantized output:

$$\dot{x}_i = x_{i+1} + f_i(y), \quad i = 1, \cdots, n-1 \tag{43}$$

$$\dot{x}_n = u + f_n(y) \tag{44}$$

$$y = x_1 \tag{45}$$

$$y^q = q(y, \mu) \tag{46}$$

where $[x_1, \cdots, x_n]^{\mathrm{T}} \in \mathbf{R}^n$ is the state, $u \in \mathbf{R}$ is the control input, $y \in \mathbf{R}$ is the output, $q(y, \mu)$ is the output quantizer with variable μ, $y^q \in \mathbf{R}$ is the quantized output available for feedback design, $[x_2, \cdots, x_n]^{\mathrm{T}}$ is the unmeasured portion of the state, and f_i's $(i = 1, \cdots, n)$ are uncertain, locally Lipschitz functions. For each $f_i(y)$, it is assumed that $|f_i(y)| \le \psi_{f_i}(|y|)$ for all y with a known $\psi_{f_i} \in \mathcal{K}_\infty$. Moreover, the quantizer $q(y, \mu)$ is assumed to satisfy property (34) with the r replaced by y and constant $M \ge 3$. As discussed before, this assumption is mild for uniform quantizers with no less than three quantization levels.

Motivated by [75], a reduced-order observer which uses the quantized output y^q can be designed as

$$\dot{\xi}_i = \xi_{i+1} + L_{i+1} y^q - L_i(\xi_2 + L_2 y^q), \quad i = 2, \cdots, n-1 \tag{47}$$

$$\dot{\xi}_n = u - L_n(\xi_2 + L_2 y^q) \tag{48}$$

where ξ_i is an estimate for the unmeasured state $x_i - L_i y$ for each $i = 2, \cdots, n$.

Define $z = [x_2 - L_2 y - \xi_2, \cdots, x_n - L_n y - \xi_n]^{\mathrm{T}}$ as the observation error. Then, the observation error system can be represented by

$$\dot{z} = g(z, y, w) \tag{49}$$

where $w = y^q - y$ represents the quantization error. By appropriately choosing the parameters of the reduced-order observer, the e_0-system can be designed to be ISS with y and w as the inputs.

Then, the quantized output-feedback control problem is transformed into a partial-state feedback control problem for the following strict-feedback system with dynamic uncertainties:

$$\dot{z} = g(z, y, w) \tag{50}$$

$$\dot{y} = \xi_2 + \phi_1(z, y) \tag{51}$$

$$\dot{\xi}_i = \xi_{i+1} + \phi_i(y, \xi_2, w), \quad i = 2, \cdots, n-1 \tag{52}$$

$$\dot{\xi}_n = u + \phi_n(y, \xi_2, w) \tag{53}$$

For system (50) ~ (53), $(y^q, \xi_2, \cdots, \xi_n)$ is available for feedback. In [64], a set-valued map design was developed to deal with the quantization. Specifically, through the design, the closed-loop quantized system can be transformed into a network of ISS subsystems. The state variables of the subsystems are defined as

$$e_0 = z \tag{54}$$

$$e_1 = y \tag{55}$$

$$e_i = \boldsymbol{d}(\xi_i, S_{i-1}(e_1, \xi_2, \cdots, \xi_{i-1}, \mu)), \quad i = 2, \cdots, n \tag{56}$$

where each $S_{i-1} : \mathbf{R} \times \cdots \times \mathbf{R} \times \mathbf{R}_+ \to \mathbf{R}$ is a well chosen set-valued map. The definition of $\boldsymbol{d}$ is given in (32). Fig. 8 shows an example of S_1 and the definition of e_2. The set-valued map with "size" depending on μ covers the influence of quantization.

Denote $e = [e_0^{\mathrm{T}}, e_1, \cdots, e_n]^{\mathrm{T}}$. Then the closed-loop quantized system can be represented by a differential inclusion

$$\dot{e} \in F(e, \mu) \tag{57}$$

If the output is within the range of the quantizer, i.e., $|y| \le M\mu$, then by using the ISS cyclic-small-gain theorem, one can find a $V : \mathbf{R}^{2n-1} \to \mathbf{R}_+$ such that

$$V(e) \ge \frac{\mu^2}{2c^2} \Rightarrow \max_{f \in F(e,\mu)} \nabla V(e) f \le -\alpha(V(e)) \tag{58}$$

where constant c can be arbitrarily chosen such that $0 < c < 1$ and α is a continuous and positive definite function. Moreover, it is guaranteed that

$$V(e) \le \frac{M^2\mu^2}{2} \Rightarrow |y| \le M\mu \tag{59}$$

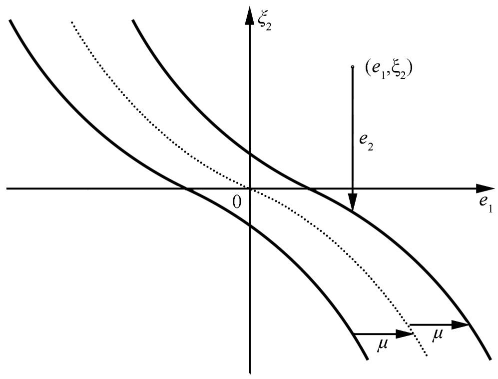

Fig. 8 The definitions of S_1 and e_2

Note that $M \ge 3$. With constant c chosen to be larger than 1/3, properties (57) and (58) together implies

$$\frac{M^2\mu^2}{2} \ge V(e) \ge \frac{\mu^2}{2c^2} \Rightarrow \max_{f \in F(e,\mu)} \nabla V(e) f \le -\alpha(V(e)) \tag{60}$$

This is in accordance with property (41).

However, since the set-valued maps depend on μ, the newly defined variables $e_2, \cdots, e_n$ depend on μ. One cannot trivially implement the standard dynamic quantization strategy to update μ during the quantized control process, as this may make $V(e)$ "jump out" of the larger invariant set $\Omega_L(\mu) = \{e : V(e) \le M^2\mu^2/2\}$. This problem was successfully solved in [64] by carefully studying the decreasing rate of $V(e)$. If an upper bound of the initial value $V(e(0))$ is known, then the zooming variable μ can be reduced slowly to realize the convergence of $V(e)$ and thus the convergence of $e_1 = y$, as shown in Fig. 9.

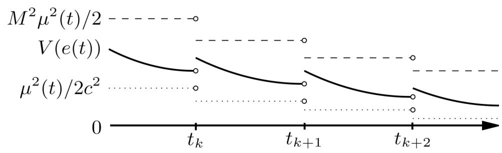

Fig. 9 The motions of $M^2\mu^2(t)/2$, $\mu^2(t)/2c^2$ and $V(e(t))$

If an upper bound of $V(e(0))$ is unknown, then a zooming-out process is needed before zooming-in. In this process, the zooming variable μ should be increased fast enough such that at some finite time t^*, $V(e(t^*)) \le M^2\mu^2(t^*)/2$. This needs some a priori knowledge on the divergence rate and observability of the system. Small-time norm-observability is an often used notion[79]. The structure of the quantized output-feedback control system proposed in [64] is shown in Fig. 10.

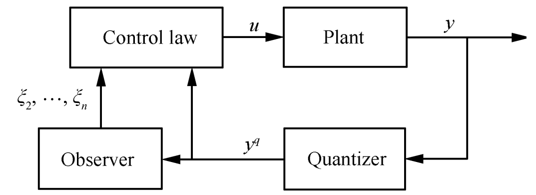

Fig. 10 A quantized output-feedback control structure

2.3 Quantized state-feedback control

If more than one quantizers are used to control a system, then the updates of the zooming variables of the quantizers should be well coordinated. In this subsection, we show how the problem is solved with ISS small-gain designs based on the result of [66]. Fig. 11 shows the quantized control structure for nonlinear systems with dynamic uncertainties. Each quantizer q_{ij} in the system has a zooming variable μ_{ij}.

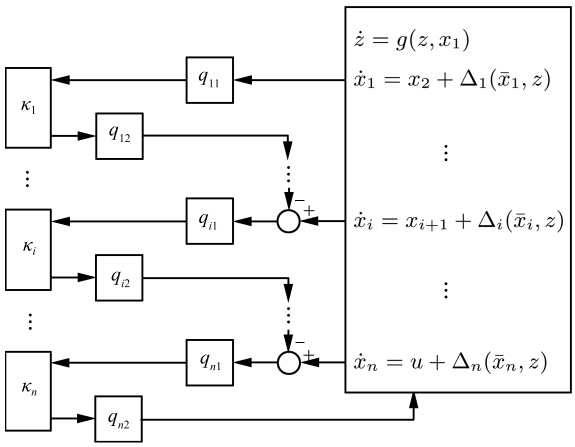

Fig. 11 A quantized control structure for high-order nonlinear systems: $x_i \in \mathbf{R}$, the z-subsystem with $z \in \mathbf{R}^m$ represents dynamic uncertainties, q_{ij} for $i = 1, \cdots, n$, $j = 1, 2$ are quantizers, and κ_i for $i = 1, \cdots, n$ are functions forming the control law

Reference [66] considers the case where the z-subsystem is ISS. A set-valued map design is employed to transform the closed-loop quantized system into a network of $n+1$ ISS subsystems with e_i $(i = 0, \cdots, n)$ as the state variables of the subsystems. Denote e as the vector of e_i for $i = 0, \cdots, n$ and μ as the vector of μ_{ij} for $i = 1, \cdots, n$, $j = 1, 2$. Then, the closed-loop quantized system can be represented by a differential inclusion $\dot{e} \in F(e, \mu)$. Moreover, the closed-loop quantized system satisfies the cyclic-small-gain condition, and one can construct a positive definite and radially unbounded V such that

$$\min_{i=1,\cdots,n, j=1,2}\{\overline{\sigma}_{ij}(\mu_{ij})\} \geq V(e) \geq \max_{i=1,\cdots,n, j=1,2}\{\underline{\sigma}_{ij}(\mu_{ij})\} \Rightarrow \max_{f \in F(e,\mu)} \nabla V(e) f \leq -\alpha(V(e)) \tag{61}$$

for all e and μ, where $\overline{\sigma}_{ij}$ and $\underline{\sigma}_{ij}$ are class $\mathcal{K}_\infty$ functions satisfying $\overline{\sigma}_{ij} > \underline{\sigma}_{ij}$, and α is a continuous and positive definite function. Intuitively, the min operator on the left-hand side of (60) is used to guarantee that all the signals to be quantized are covered by the ranges of their corresponding quantizers. Clearly, property (60) defines the nested invariant sets for the closed-loop quantized system.

In the zooming-in stage, to always satisfy the condition of the implication in (60), the zooming variables μ_{ij} should be updated cooperatively. One solution is to adjust μ_{ij} such that for all $i = 1, \cdots, n$, $j = 1, 2$,

$$\overline{\sigma}_{ij}(\mu_{ij}(t)) = \theta(t) \tag{62}$$

for all $t \geq 0$. Then, property (60) implies

$$\theta \geq V(e) \geq \chi(\theta) \Rightarrow \max_{f \in F(e,\mu)} \nabla V(e) f \leq -\alpha(V(e)) \tag{63}$$

where χ is a class $\mathcal{K}_\infty$ function being less than the identity function. With this treatment, the dynamic quantization problem can be solved by designing an update law for θ. It should be noted that due to the set-valued map design, $e_2, \cdots, e_n$ depend on the zooming variables μ_{ij} and thus θ. This problem can be solved similarly as for quantized output-feedback control discussed in Subsection 2.2.

3 Conclusions and open problems

This paper provides a survey on the state of the art of quantized feedback control for nonlinear systems. As argued in the paper, quantized nonlinear control is strongly connected to robust nonlinear control. And advanced robust control design methods, such as the ISS small-gain approach, are powerful in handling the new problems caused by quantization in nonlinear control. Expectedly, the preliminary results presented in the paper can be further generalized in several directions, in view of the rich literature of nonlinear control over the last three decades. It should be mentioned that there are more open problems in this field than the available results. Some open problems of great interest are stated below:

1) Geometric nonlinear control with quantized signals. The classical yet important topic of controllability and observability for nonlinear systems needs to be revisited[80−81], when only quantized signals are allowed. In addition, the relationships between controllability and stabilizability[82], and feedback linearization theory[80], need to be revisited as well in the context of quantized feedback control.

2) Tracking via quantized feedback. While this paper and the work of others focus on quantized stabilization, the problem of quantized feedback tracking is of more practical interest and recovers stabilization as a special case. Instead of forcing the state or the output to the origin or a set-point of interest, the quantized feedback tracking problem seeks a quantized feedback controller so that the output follows a desirable reference signal or the state follows the desired state of a reference model. This problem has received practically no attention in the present literature. Closely related to this problem is the output regulation theory[83−84] that consists of searching for (unquantized) feedback control laws to achieve asymptotic tracking with disturbance rejection, when the disturbance and reference signals are generated by an exo-system. The well-known internal model principle serves as a bridge to convert the output regulation problem into a stabilization problem for a transformed system. To what extent will the internal model principle remain valid and applicable when only quantized signals are used?

3) Decentralized quantized control. In the decentralized control setting, local controllers are used to control the subsystems of a large-scale system[85]. Among the main characteristics of decentralized control are the dramatic reduction of computational complexity and the enhancement of robustness against uncertain interactions or loss of interaction. The ISS small-gain designs for decentralized control[75, 86] makes it possible to further take into account the effect of quantization. It should be noted that the decentralized measurement feedback control problem, which is closely related to decentralized quantized control, has been studied in a recent paper[87]. For dynamic quantization, the zooming variables of the quantizers of different subsystems should coordinate with each other. This is still the case when decentralized control is reduced to centralized control, as shown in Subsection 2.3. Another problem with decentralized dynamic quantization is that the updates of the zooming variables of the quantizers may not be synchronized. For such a problem, the small-gain results for hybrid system[76, 88−89] should be helpful.

4) Quantized adaptive control. Controllers are expected to possess adaptive capabilities to cope with "large" uncertainties. A further extension of the previously developed methodology to quantized adaptive control is of practical interest for engineering applications. The recent achievements[90−91] provide a basis for future research in this direction. Reference [90] proposes a Lyapunov-based framework for adaptive quantized control of linear uncertain systems modeled in discrete-time. In [91], a direct adaptive control strategy was developed for nonlinear uncertain systems with input quantizers under the assumption that the system is robustly stabilizable with respect to sector bounded uncertainties.

5) Networked and quantized control systems with time delays. In modern networked control systems, data transmission through communication channels inevitably results in time delays, a severe cause for poor performance and even instability of the system in question. Data-sampling may be considered as a special case of time-delay. Reference [92] studied the connections between the ISS small-gain theorem and the Razumikhin theorem, in the context of time-delay nonlinear systems. Recently, a necessary and sufficient condition for robust stabilizability of nonlinear, time-varying systems with delayed state measurements was presented in [61], which is a new framework capable of dealing with quantization and time-delay, at the same time is of paramount importance for the transition of advanced nonlinear control theory to practice.

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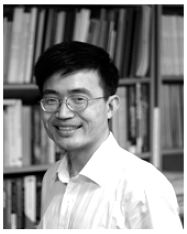

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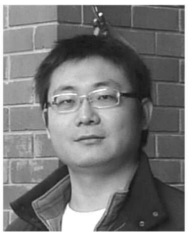

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