Mean-Square Exponential Input-to-State Stability of Numerical Solutions for Stochastic Control Systems

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Abstract This paper deals with the mean-square exponential input-to-state stability (exp-ISS) of numerical solutions for stochastic control systems (SCSs). Firstly, it is shown that a finite-time strong convergence condition holds for the stochastic θ-method on SCSs. Then, we can see that the mean-square exp-ISS of an SCS holds if and only if that of the stochastic θ-method (for sufficiently small step sizes) is preserved under the finite-time strong convergence condition. Secondly, for a class of SCSs with a one-sided Lipschitz drift, it is proved that two implicit Euler methods (for any step sizes) can inherit the mean-square exp-ISS property of the SCSs. Finally, numerical examples confirm the correctness of the theorems presented in this study.

Key words Mean-square exponential input-to-state stability, stochastic control systems, stochastic θ-method, strong convergence

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The well-known input-to-state stability (ISS) property of deterministic systems originated in [1] and was investigated quite intensively in recent years (see e.g. [2−9]). Especially, some concepts of exp-ISS for SCSs have appeared in [10−12]. These concepts are extensions of ISS for deterministic systems.

The exp-ISS of an SCS usually depends on the existence of an appropriate control Lyapunov function. However, in general, there is no effective method to find such control Lyapunov function. This absence drives us to carry out careful numerical simulations using a numerical method with a small step-size Δ. Then, for the mean-square exp-ISS, two key questions are raised.

Question 1. If the SCS satisfies mean-square exp-ISS, can the numerical method preserve the mean-square exp-ISS for sufficiently small Δ?

Question 2. If the numerical method satisfies mean-square exp-ISS for small Δ, can we infer that the underlying SCS also satisfies mean-square exp-ISS?

Results on answer of Question 1 and Question 2 for uncontrolled stochastic systems can be found in [13−16]. The input-to-state stability of Runge-Kutta methods for deterministic control systems was investigated in [17]. Furthermore, the input-to-state stability of Euler-Maruyama method for SCSs was investigated in [18]. However, there appears to be still a large gap calling for further research on the mean-square exp-ISS of stochastic high-order numerical methods for SCSs. In addition, it is of great importance to study control problems by using numerical methods. Since the research on control problems (e.g., ISS) becomes much more difficult by using only conventional methods with respect to the rapid development of some larger engineering designs and so on.

Our aim in this study is to give definitely positive answers to both Question 1 and Question 2 without an appropriate control Lyapunov function by drawing on the ideas in [13−16]. The organization of this paper is as follows. In Section 1, we give our definitions of mean-square exp-ISS for SCSs and numerical methods. In Section 2, we show that a required finite-time strong convergence condition holds for the stochastic θ-method on SCSs with globally Lipschitz coefficients and mean-square continuous random inputs. On this basis, we show that the mean-square exp-ISS of the SCSs is equivalent to that of the stochastic θ-methods for sufficiently small step-sizes. In Section 3, we show that two implicit Euler methods can inherit mean-square exp-ISS property for SCSs under a one-sided Lipschitz condition on the drift. In Section 4, two numerical examples are given to show the correctness of these theorems.

Notations. Let $\mathbf{R}^n$ denote the n-dimensional Euclidean space, $\mathbf{R}^{n\times m}$ is the set of all $n\times m$ real matrices and $\mathbf{Z}$ is the set of all integers. The quadruplet $(\Omega,\mathfrak{F},\{\mathfrak{F}_t\}_{t\geq 0},\mathcal{P})$ is a complete probability space with a filtration $\{\mathfrak{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., it is right continuous and $\mathfrak{F}_0$ contains all $\mathcal{P}$-null sets). The symbol $\mathrm{E}\{\cdot\}$ stands for the mathematical expectation. Let $|\cdot|$ denote both the Euclidean norm in $\mathbf{R}^n$ and the trace norm in $\mathbf{R}^{n\times m}$. Denote by $L^2_{\mathfrak{F}_t}(\Omega;\mathbf{R}^n)$ the family of all $\mathfrak{F}_t$-measurable random variables $\boldsymbol{\xi}:\Omega\rightarrow\mathbf{R}^n$ such that $\mathrm{E}|\boldsymbol{\xi}|^2<\infty$. A function $\gamma:\mathbf{R}^+\rightarrow\mathbf{R}^+$ is called a κ-function if it is continuous, strictly increasing and $\gamma(0)=0$.

1 The mean-square exp-ISS

Consider the following n-dimensional Itô SCS:

$$\mathrm{d}\boldsymbol{y}(t)=\boldsymbol{f}(\boldsymbol{y}(t),\boldsymbol{u}(t))\mathrm{d}t+\boldsymbol{g}(\boldsymbol{y}(t),\boldsymbol{u}(t))\mathrm{d}w(t),\quad t\geq 0 \tag{1}$$

where $\boldsymbol{y}(t)\in\mathbf{R}^n,\boldsymbol{u}(t)\in\mathbf{R}^m$ are the state and the input vectors of the system, respectively. The standard p-dimensional Wiener process is denoted by $w(t)$. The set of admissible inputs, which is denoted by $\mathcal{F}(\mathbf{R}^m)$, is the set of all progressively measurable random functions $\boldsymbol{u}:\Omega\times[0,\infty)\rightarrow\mathbf{R}^m$ such that the supremum norm $|\boldsymbol{u}|_{\sup}=\sup\{|\boldsymbol{u}(t)|,t\geq 0,\text{a.s.}\}$ remains finite. Therefore, for every $t\geq 0$, the random input $\boldsymbol{u}$ is $\mathfrak{F}_t\times B_t$-measurable (B_t is the σ-algebra of Borel subsets of $[0,t]$). As a direct consequence each $\boldsymbol{u}(t)$ is $\mathfrak{F}_t$-adapted.

We always assume that $\boldsymbol{f}:\mathbf{R}^n\times\mathbf{R}^m\rightarrow\mathbf{R}^n$ and $\boldsymbol{g}:\mathbf{R}^n\times\mathbf{R}^m\rightarrow\mathbf{R}^{n\times p}$ are both Borel measurable such that the SCS (1) has a unique solution for all $t\geq 0$, namely, for every $\boldsymbol{\xi}\in L^2_{\mathfrak{F}_0}(\Omega;\mathbf{R}^n)$ and input $\boldsymbol{u}(t)\in\mathcal{F}(\mathbf{R}^m)$, there exists a unique solution $\boldsymbol{y}(t;0,\boldsymbol{\xi},\boldsymbol{u}(t))$ of SCS (1) starting from $\boldsymbol{\xi}$ at $t=0$. Throughout the present paper, we assume

$$\boldsymbol{f}(0,0)=0,\qquad \boldsymbol{g}(0,0)=0 \tag{2}$$

for the study of the equilibrium solution of (1).

Definition 1. SCS (1) is said to have the mean-square exponential input-to-state stability (exp-ISS in short) if there exists a κ-function β and positive constants M and λ such that, for all initial data $\boldsymbol{\xi}\in L^2_{\mathfrak{F}_0}(\Omega;\mathbf{R}^n)$ and for all random inputs $\boldsymbol{u}(t)\in\mathcal{F}(\mathbf{R}^m)$,

$$\mathrm{E}|\boldsymbol{y}(t)|^2\leq M\mathrm{E}|\boldsymbol{\xi}|^2\mathrm{e}^{-\lambda t}+\mathrm{E}\beta(|\boldsymbol{u}|^2_{\sup}),\quad t\geq 0 \tag{3}$$

We refer to λ as a rate constant, M as a growth constant and β as a gain function.

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We suppose that for SCS (1) a numerical method is available with a given positive step-size Δ and its discrete approximations are denoted by $\boldsymbol{x}_k \approx \boldsymbol{y}(k\Delta)(\boldsymbol{x}_0 = \boldsymbol{\xi})$. We also suppose that there is a well-defined interpolation process that extends the discrete approximation $\{\boldsymbol{x}_k\}_{k\in\mathbf{Z}^+}$ to a continuous time approximation $\{\boldsymbol{x}(t)\}_{t\in\mathbf{R}^+}$ with $\boldsymbol{x}(k\Delta) = \boldsymbol{x}_k$. Parallel to Definition 1, we define the mean-square exp-ISS for a numerical method.

Definition 2. For a given positive step-size Δ, a numerical method is said to have the mean-square exp-ISS on the SCS (1), if there exists a κ-function γ and positive constants N and l such that, for all initial data $\boldsymbol{\xi} \in L^2_{\mathfrak{F}_0}(\Omega; \mathbf{R}^n)$ and for all random inputs $\boldsymbol{u}(t) \in \mathcal{F}(\mathbf{R}^m)$,

$$\mathrm{E}|\boldsymbol{x}(t)|^2 \le N\mathrm{E}|\boldsymbol{\xi}|^2\mathrm{e}^{-lt} + \mathrm{E}\gamma(|\boldsymbol{u}|^2_{\sup}), \quad t \ge 0 \tag{4}$$

We refer to l as a rate constant, N as a growth constant and γ as a gain function.

2 Stochastic $\boldsymbol{\theta}$-methods

In this section, the following finite-time strong convergence condition plays an important role in the numerical method sharing the mean-square exp-ISS with the SCS (1).

Condition 1. For all sufficiently small Δ, the numerical method applied to SCS (1) with initial condition $\boldsymbol{x}_0 = \boldsymbol{y}(0) = \boldsymbol{\xi}$ and random input $u(t) \in \mathcal{F}(\mathbf{R}^m)$ satisfies, for any $T > 0$,

$$\sup_{0\le t\le T} \mathrm{E}|\boldsymbol{x}(t)|^2 \le B_{\boldsymbol{\xi},T,|\boldsymbol{u}|_{\sup}} \tag{5}$$

where $B_{\boldsymbol{\xi},T,|\boldsymbol{u}|_{\sup}}$ depends on $\boldsymbol{\xi}, T$ and $|\boldsymbol{u}|_{\sup}$, but not upon Δ, and

$$\sup_{0\le t\le T} \mathrm{E}|\boldsymbol{x}(t)-\boldsymbol{y}(t)|^2 \le C_T\Delta(\sup_{0\le t\le T} \mathrm{E}|\boldsymbol{x}(t)|^2)+D_T\Delta(\mathrm{E}|\boldsymbol{u}|^2_{\sup}) \tag{6}$$

where C_T and D_T depend on T but not on $\boldsymbol{\xi}, \Delta$ and $|u|_{\sup}$.

Lemma 1. Suppose that a numerical method satisfies Condition 1. Then the SCS (1) satisfies the mean-square exp-ISS if and only if there exists a $\Delta > 0$ such that the numerical method satisfies the mean-square exp-ISS with rate constant l, growth constant N, gain γ, step-size Δ and global constants C_T, D_T for $T := 1 + (4\log N)/l$ satisfying conditions

$$C_{2T}\mathrm{e}^{lT}(\Delta + \sqrt{\Delta}) + 1 + \sqrt{\Delta} \le \mathrm{e}^{(\frac{1}{4})lT}, \quad C_T\Delta \le 1 \tag{7}$$

and

$$(D_{2T} + C_{2T}\gamma)(\Delta + \sqrt{\Delta}) + \gamma\sqrt{\Delta} \le 1, \quad (D_T + C_T\gamma)\Delta \le 1 \tag{8}$$

Proof. Similar to the proof of Theorem 3 in [18], we can easily see that the assertion in lemma holds for any numerical methods satisfying Condition 1. $\square$

Clearly, from Lemma 1, if we want to show the mean-square exp-ISS of stochastic θ-methods equivalent to that of SCS (1), we only need to show stochastic θ-methods satisfy Condition 1.

Consider the following stochastic θ-method

$$\begin{aligned}\boldsymbol{x}_{k+1} = \ & \boldsymbol{x}_k + (1-\theta)\boldsymbol{f}(\boldsymbol{x}_k, \boldsymbol{u}_k)\Delta + \\ & \theta\boldsymbol{f}(\boldsymbol{x}_{k+1}, \boldsymbol{u}_{k+1})\Delta + \boldsymbol{g}(\boldsymbol{x}_k, \boldsymbol{u}_k)\Delta w_k\end{aligned} \tag{9}$$

where $\boldsymbol{x}_k = \boldsymbol{x}(k\Delta), \Delta w_k = w((k+1)\Delta) - w(k\Delta), \boldsymbol{u}_k = \boldsymbol{u}(k\Delta)$ and $\theta \in [0,1]$ is a free parameter. The choice $\theta = 0$ and 1 in (9) derives the widely-used Euler-Maruyama method and the drift-implicit Euler method (19), respectively. For the discrete approximations by (9), we introduce the continuous approximation given by

$$\begin{aligned}\boldsymbol{x}(t) = \ & \boldsymbol{\xi} + \int_0^t [(1-\theta)\boldsymbol{f}(z_1(s), U_1(s)) + \\ & \theta\boldsymbol{f}(z_2(s), U_2(s))]\mathrm{d}s + \int_0^t g(z_1(s), U_1(s))\mathrm{d}w(s)\end{aligned} \tag{10}$$

where

$$z_1(t) = \sum_{k=0}^{\infty} \boldsymbol{x}_k \mathbf{1}_{[k\Delta,(k+1)\Delta)}(t)$$

$$z_2(t) = \sum_{k=0}^{\infty} \boldsymbol{x}_{k+1} \mathbf{1}_{[k\Delta,(k+1)\Delta)}(t)$$

$$U_1(t) = \sum_{k=0}^{\infty} u_k \mathbf{1}_{[k\Delta,(k+1)\Delta)}(t)$$

$$U_2(t) = \sum_{k=0}^{\infty} u_{k+1} \mathbf{1}_{[k\Delta,(k+1)\Delta)}(t)$$

here $\mathbf{1}_G$ denotes the indicator function for the set G.

Assumption 1. Assume that both f and g are globally Lipschitz continuous, that is,

$$|\boldsymbol{f}(\boldsymbol{y}, \boldsymbol{u}) - \boldsymbol{f}(\bar{\boldsymbol{y}}, \bar{\boldsymbol{u}})|^2 \le K_1(|\boldsymbol{y} - \bar{\boldsymbol{y}}|^2 + |\boldsymbol{u} - \bar{\boldsymbol{u}}|^2) \tag{11}$$

and

$$|\boldsymbol{g}(\boldsymbol{y}, \boldsymbol{u}) - \boldsymbol{g}(\bar{\boldsymbol{y}}, \bar{\boldsymbol{u}})|^2 \le K_2(|\boldsymbol{y} - \bar{\boldsymbol{y}}|^2 + |\boldsymbol{u} - \bar{\boldsymbol{u}}|^2) \tag{12}$$

for all $\boldsymbol{y}, \bar{\boldsymbol{y}} \in \mathbf{R}^n$ and $\boldsymbol{u}, \bar{\boldsymbol{u}} \in \mathcal{F}(\mathbf{R}^m)$, where K_1 and K_2 are positive constants. Furthermore, we also assume that

$$\mathrm{E}|\boldsymbol{u}(t+\Delta) - \boldsymbol{u}(t)|^2 \le L\Delta^2\mathrm{E}|\boldsymbol{u}|^2_{\sup} \ (L > 0) \tag{13}$$

for all sufficiently small $\Delta > 0$, $t \in \mathbf{R}^+$ and $\boldsymbol{u}(t) \in \mathcal{F}(\mathbf{R}^m)$. It implies that the random input $\boldsymbol{u}(t)$ is mean-square continuous.

The proof of following Lemmas $2\sim 4$ are similar to that of Lemmas A.1 $\sim$ A.3 in [13], respectively.

Lemma 2. Under the global Lipschtiz conditions (11) and (12), if $K_1\theta\Delta < 1$, the equation (9) can be solved uniquely for $\boldsymbol{x}_{k+1}$, with probability 1. See Lemma A.1 in [13].

Lemma 3. Under the global Lipschtiz conditions (11) and (12), for sufficiently small Δ, the discrete stochastic θ-method solution (9) satisfies

$$\mathrm{E}|\boldsymbol{x}_{k+1}|^2 \le 2\mathrm{E}|\boldsymbol{x}_k|^2 + (6K_2 + 1)\Delta\mathrm{E}|\boldsymbol{u}|^2_{\sup}$$

for all $k \ge 0$. See Lemma A.2 in [13].

Lemma 4. Under the global Lipschtiz conditions (11) and (12), for sufficiently small Δ, the stochastic θ-method solution (10) satisfies

$$\sup_{0\le t\le T} \mathrm{E}|\boldsymbol{x}(t)|^2 < \infty \tag{14}$$

and

$$\begin{aligned}&\sup_{0\le t\le T} \{\mathrm{E}|\boldsymbol{x}(t) - z_1(t)|^2 \vee \mathrm{E}|\boldsymbol{x}(t) - z_2(t)|^2\} \le \\ &(2K_2 + 1)\Delta \sup_{0\le t\le T} \mathrm{E}|\boldsymbol{x}(t)|^2 + (2K_2 + 1)\Delta\mathrm{E}|\boldsymbol{u}|^2_{\sup}\end{aligned} \tag{15}$$

for all $T > 0$. See Lemma A.3 in [13].

Now we are ready to state our goal.

Theorem 1. Under Assumption 1, for sufficiently small Δ, the stochastic θ-method solution (10) satisfies

$$\sup_{0\le t\le T} \mathrm{E}|\boldsymbol{x}(t)-\boldsymbol{y}(t)|^2 \le C_T\Delta \sup_{0\le t\le T} \mathrm{E}|\boldsymbol{x}(t))|^2 + D_T\Delta \mathrm{E}|\boldsymbol{u}|^2_{\sup} \tag{16}$$

where

$$C_T = 4T(2K_1T+K_2)(2K_2+1)\mathrm{e}^{4T(2K_1T+K_2)}$$

and

$$D_T = 2T(2K_1T+K_2)(4K_2+3)\mathrm{e}^{4T(2K_1T+K_2)}$$

which are independent of Δ. This, together with Lemma 4, shows that under Assumption 1, the stochastic θ-method satisfies Condition 1.

Proof. It follows from (1) and (10) that for any $0\le t\le T$,

$$\begin{aligned}\boldsymbol{x}(t)-\boldsymbol{y}(t) = & \int_0^t ((1-\theta)[\boldsymbol{f}(z_1(s),U_1(s))-\boldsymbol{f}(\boldsymbol{y}(s),\boldsymbol{u}(s))] + \\ & \theta[\boldsymbol{f}(z_2(s),U_2(s)-\boldsymbol{f}(\boldsymbol{y}(s),\boldsymbol{u}(s)])\,\mathrm{d}s + \\ & \int_0^t (\boldsymbol{g}(z_1(s),U_1(s))-\boldsymbol{g}(\boldsymbol{y}(s),\boldsymbol{u}(s)))\,\mathrm{d}w(s)\end{aligned}$$

Hence, for sufficiently small Δ, the above equation yields

$$\begin{aligned}&\mathrm{E}|\boldsymbol{x}(t)-\boldsymbol{y}(t)|^2 \le 2K_1TE\int_0^t (|z_1(s)-\boldsymbol{y}(s)|^2 + \\ &|U_1(s)-\boldsymbol{u}(s)|^2 + |z_2(s)-\boldsymbol{y}(s)|^2 + |U_2(s)- \\ &\boldsymbol{u}(s)|^2)\mathrm{d}s + 2K_2\mathrm{E}\int_0^t (|z_1(s)-\boldsymbol{y}(s)|^2 + |U_1(s)- \\ &\boldsymbol{u}(s)|^2)\mathrm{d}s \le 4T(2K_1T+K_2)(2K_2+1)\Delta \\ &(\sup_{0\le t\le T}\mathrm{E}|\boldsymbol{x}(t)|^2 + \mathrm{E}|\boldsymbol{u}|^2_{\sup}) + \\ &4(2K_1T+K_2)\mathrm{E}\int_0^t |\boldsymbol{x}(s)-\boldsymbol{y}(s)|^2\mathrm{d}s + \\ &2T(2K_1T+K_2)L\Delta^2\mathrm{E}|\boldsymbol{u}|^2_{\sup} \le \\ &4(2K_1T+K_2)\mathrm{E}\int_0^t |\boldsymbol{x}(s)-\boldsymbol{y}(s)|^2\mathrm{d}s + \\ &4T(2K_1T+K_2)(2K_2+1)\Delta \sup_{0\le t\le T}\mathrm{E}|\boldsymbol{x}(t)|^2 + \\ &2T(2K_1T+K_2)(4K_2+3)\Delta\mathrm{E}|\boldsymbol{u}|^2_{\sup}\end{aligned} \tag{17}$$

if $L\Delta\le 1$. An application of the continuous Gronwall Lemma yields a bound of the form

$$\mathrm{E}|\boldsymbol{x}(t)-\boldsymbol{y}(t)|^2 \le C_T\Delta \sup_{0\le t\le T}\mathrm{E}|\boldsymbol{x}(t))|^2 + D_T\Delta\mathrm{E}|\boldsymbol{u}|^2_{\sup}$$

Since this holds for any $t\in[0,T]$, the assertion (16) must hold. □

From Condition 1, Lemma 1, and Theorem 1, we can easily get Theorem 2.

Theorem 2. Under Assumption 1, the SCS (1) satisfies the mean-square exp-ISS if and only if there exists a $\Delta>0$ such that the stochastic θ-method (9) satisfies the mean-square exp-ISS.

3 Mean-square exp-ISS under a one-sided Lipschitz condition

In this section, we show that two types of implicit Euler methods can inherit the mean-square exp-ISS property of SCSs that have a one-sided Lipschitz drift and globally Lipschitz diffusion terms. Therefore we put forth the following.

Assumption 2. There exist positive constants μ,ν and c with $2\mu-c>0$ such that

$$\begin{aligned}&\langle \boldsymbol{y}-\bar{\boldsymbol{y}}, \boldsymbol{f}(\boldsymbol{y},\boldsymbol{u})-f(\bar{\boldsymbol{y}},\bar{\boldsymbol{u}})\rangle \le -\mu|\boldsymbol{y}-\bar{\boldsymbol{y}}|^2+\nu|\boldsymbol{u}-\bar{\boldsymbol{u}}|^2 \\ &|g(\boldsymbol{y},\boldsymbol{u})-g(\bar{\boldsymbol{y}},\bar{\boldsymbol{u}})|^2 \le c(|\boldsymbol{y}-\bar{\boldsymbol{y}}|^2+|\boldsymbol{u}-\bar{\boldsymbol{u}}|^2)\end{aligned} \tag{18}$$

for all $\boldsymbol{y},\bar{\boldsymbol{y}}\in\mathbf{R}^n, u,\bar{\boldsymbol{u}}\in\mathcal{F}(\mathbf{R}^m)$.

The inequality (17), which can be said to be one-sided Lipschitz condition, has been frequently referred in the literature and plays a useful role in the stability analysis of uncontrolled stochastic systems[13−14] and deterministic systems[19], especially in the stiff case.

Suppose that $\boldsymbol{y}(t)$ and $\bar{\boldsymbol{y}}(t)$ are the exact solutions of SCS (1) with the initial data $\boldsymbol{y}(0)$ and $\bar{\boldsymbol{y}}(0)$ and the random inputs $\boldsymbol{u}(t)$ and $\bar{\boldsymbol{u}}(t)$, respectively.

The corresponding numerical solutions are denoted by $\boldsymbol{x}_k$ and $\bar{\boldsymbol{x}}_k$ so as to satisfy $\boldsymbol{x}_k\approx\boldsymbol{y}(k\Delta)$ ($\boldsymbol{x}_0=\boldsymbol{y}(0)$) and $\bar{\boldsymbol{x}}_k\approx\bar{\boldsymbol{y}}(k\Delta)$ ($\bar{\boldsymbol{x}}_0=\bar{\boldsymbol{y}}$ (0)), respectively.

The following theorem provides a ground for study of the mean-square exp-ISS of numerical methods for SCS (1).

Theorem 3. Under Assumption 2, any two solutions to SCS (1) satisfy

$$\mathrm{E}|\boldsymbol{y}(t)-\bar{\boldsymbol{y}}(t)|^2 \le \mathrm{E}|\boldsymbol{y}(0)-\bar{\boldsymbol{y}}(0)|^2\mathrm{e}^{-(2\mu-c)t}+\frac{2\nu+c}{2\mu-c}\mathrm{E}|\boldsymbol{u}-\bar{\boldsymbol{u}}|^2_{\sup}$$

and the SCS (1) possesses the mean-square exp-ISS with rate constant $\lambda=2\mu-c$, growth constant $M=1$ and gain $\beta=(2\nu+c)/(2\mu-c)$.

Proof. An application of the Itô formula to $|\boldsymbol{y}(t)-\bar{\boldsymbol{y}}(t)|^2$ implies

$$\begin{aligned}d|\boldsymbol{y}-\bar{\boldsymbol{y}}|^2 \le\ & (2\langle\boldsymbol{f}(\boldsymbol{y},\boldsymbol{u})-\boldsymbol{f}(\bar{\boldsymbol{y}},\bar{\boldsymbol{u}}),\boldsymbol{y}-\bar{\boldsymbol{y}}\rangle + \\ & |\boldsymbol{g}(\boldsymbol{y},\boldsymbol{u})-\boldsymbol{g}(\bar{\boldsymbol{y}},\bar{\boldsymbol{u}})|^2)\mathrm{d}t + \mathrm{d}M(\boldsymbol{y},\bar{\boldsymbol{y}},\boldsymbol{u},\bar{\boldsymbol{u}})\end{aligned}$$

where $M(\boldsymbol{y},\bar{\boldsymbol{y}},\boldsymbol{u},\bar{\boldsymbol{u}}) = 2\int_0^t\langle\boldsymbol{g}(\boldsymbol{y}(s),\boldsymbol{u}(s)-\boldsymbol{g}(\bar{\boldsymbol{y}}(s),\bar{\boldsymbol{u}}(s)),\boldsymbol{y}(s)-\bar{\boldsymbol{y}}(s)\rangle\mathrm{d}w(s)$ is a martingale. Under Assumption 2, an expectation operation after integration gives

$$\begin{aligned}&\mathrm{E}|\boldsymbol{y}(t)-\bar{\boldsymbol{y}}(t)|^2 - \mathrm{E}|\boldsymbol{y}(0)-\bar{\boldsymbol{y}}(0)|^2 \le \\ &-(2\mu-c)\mathrm{E}\int_0^t |\boldsymbol{y}(s)-\bar{\boldsymbol{y}}(s)|^2\mathrm{d}s + \\ &(2\nu+c)\mathrm{E}\int_0^t |\boldsymbol{u}(s)-\bar{\boldsymbol{u}}(s)|^2\mathrm{d}s \le \\ &-(2\mu-c)\mathrm{E}\int_0^t |\boldsymbol{y}(s)-\bar{\boldsymbol{y}}(s)|^2\mathrm{d}s + \\ &(2\nu+c)\mathrm{E}\int_0^t \mathrm{e}^{(2\mu-c)(t-s)}|\boldsymbol{u}(s)-\bar{\boldsymbol{u}}(s)|^2\mathrm{d}s \le \\ &-(2\mu-c)\mathrm{E}\int_0^t |\boldsymbol{y}(s)-\bar{\boldsymbol{y}}(s)|^2\mathrm{d}s + \\ &(2\nu+c)\frac{\mathrm{e}^{(2\mu-c)t}-1}{2\mu-c}\mathrm{E}|\boldsymbol{u}-\bar{\boldsymbol{u}}|^2_{\sup}\end{aligned}$$

Then, from the Gronwall inequality, we have

$$\begin{aligned}&\mathrm{E}|\boldsymbol{y}(t)-\bar{\boldsymbol{y}}(t)|^2 \le (\mathrm{E}|\boldsymbol{y}(0)-\bar{\boldsymbol{y}}(0)|^2 + \\ &(2\nu+c)\frac{\mathrm{e}^{(2\mu-c)t}-1}{2\mu-c}\mathrm{E}|\boldsymbol{u}-\bar{\boldsymbol{u}}|^2_{\sup})\mathrm{e}^{-(2\mu-c)t} \le \\ &\mathrm{e}^{-(2\mu-c)t}\mathrm{E}|\boldsymbol{y}(0)-\bar{\boldsymbol{y}}(0)|^2 + \frac{2\nu+c}{2\mu-c}\mathrm{E}|\boldsymbol{u}-\bar{\boldsymbol{u}}|^2_{\sup}\end{aligned}$$

Since $\bar{\boldsymbol{y}}(t) \equiv 0$ is a solution of the SCS (1) with the initial data $\bar{\boldsymbol{y}}(0) \equiv 0$ and the input $\bar{\boldsymbol{u}}(t) \equiv 0$, the mean-square exp-ISS of the SCS (1) follows immediately. □

As stated in Lemma 4.1 in [13], the Euler-Maruyama method provides only a poor approximation of the solution, particulary in the case where the coefficients are not globally Lipschitz. This is a similar phenomenon in the deterministic Euler case. Thus, in this section, we consider the following two types of implicit Euler methods[13, 20].

1) The drift-implicit Euler method:

$$\boldsymbol{x}_{k+1} = \boldsymbol{x}_k + \boldsymbol{f}(\boldsymbol{x}_{k+1}, \boldsymbol{u}_{k+1})\Delta + \boldsymbol{g}(\boldsymbol{x}_k, \boldsymbol{u}_k)\Delta w_k \tag{19}$$

2) The split-step backward Euler methods:

$$\begin{aligned} \boldsymbol{x}_k^* &= \boldsymbol{x}_k + \boldsymbol{f}(\boldsymbol{x}_k^*, \boldsymbol{u}_k)\Delta \\ \boldsymbol{x}_{k+1} &= \boldsymbol{x}_k^* + \boldsymbol{g}(\boldsymbol{x}_k^*, \boldsymbol{u}_k)\Delta w_k \end{aligned} \tag{20}$$

Here the increment of the Wiener process Δw_k means $w((k+1)\Delta) - w(k\Delta)$, $\boldsymbol{x}_k = \boldsymbol{x}(k\Delta)$ and $\boldsymbol{u}_k = \boldsymbol{u}(k\Delta)$.

The following lemma is partially similar to Lemma 4.3 of [13] and Lemma 3.4 of [20]. Hence we skip its proof.

Lemma 5. Under Assumption 2, given $\boldsymbol{b}^{(1)}, \boldsymbol{b}^{(2)} \in \mathbf{R}^n$, random inputs $\boldsymbol{u}^{(1)}, \boldsymbol{u}^{(2)} \in \mathcal{F}(\mathbf{R}^m)$ and positive Δ, let $\boldsymbol{a}^{(1)}, \boldsymbol{a}^{(2)} \in \mathbf{R}^n$ satisfy the implicit equation

$$\boldsymbol{a}^{(i)} - \Delta \boldsymbol{f}(\boldsymbol{a}^{(i)}, \boldsymbol{u}^{(i)}) = \boldsymbol{b}^{(i)}, \quad i = 1, 2$$

Then, the solutions $\boldsymbol{a}^{(1)}, \boldsymbol{a}^{(2)}$ satisfy

$$(1+2\mu\Delta)|\boldsymbol{a}^{(1)} - \boldsymbol{a}^{(2)}|^2 \le |\boldsymbol{b}^{(1)} - \boldsymbol{b}^{(2)}|^2 + 2\nu|\boldsymbol{u}^{(1)} - \boldsymbol{u}^{(2)}|^2 \tag{21}$$

Lemma 6. Under Assumption 2, any two solutions of the drift-implicit Euler method or the split-step backward Euler method satisfy

$$\mathrm{E}|\boldsymbol{x}_{k+1} - \bar{\boldsymbol{x}}_{k+1}|^2 \le \frac{1+c\Delta}{1+2\mu\Delta}\mathrm{E}|\boldsymbol{x}_k - \bar{\boldsymbol{x}}_k|^2 + \frac{2\nu + c\Delta}{1+2\mu\Delta}\mathrm{E}|\boldsymbol{u} - \bar{\boldsymbol{u}}|^2_{\sup} \tag{22}$$

and any solution satisfies

$$\mathrm{E}|\boldsymbol{x}_{k+1}|^2 \le \frac{1+c\Delta}{1+2\mu\Delta}\mathrm{E}|\boldsymbol{x}_k|^2 + \frac{2\nu + c\Delta}{1+2\mu\Delta}\mathrm{E}|\boldsymbol{u}|^2_{\sup} \tag{23}$$

Proof. For the drift-implicit Euler method (19), Lemma 5 gives

$$\begin{aligned} (1+2\mu\Delta)|\boldsymbol{x}_{k+1} - \bar{\boldsymbol{x}}_{k+1}|^2 \le{}& |\boldsymbol{x}_k - \bar{\boldsymbol{x}}_k + (\boldsymbol{g}(\boldsymbol{x}_k, u_k) - \\ & \boldsymbol{g}(\bar{\boldsymbol{x}}_k, \bar{u}_k))\Delta w_k|^2 + 2\nu|\boldsymbol{u}_k - \bar{\boldsymbol{u}}_k|^2 \le \\ & |\boldsymbol{x}_k - \bar{\boldsymbol{x}}_k|^2 + |(\boldsymbol{g}(\boldsymbol{x}_k, \boldsymbol{u}_k) - \boldsymbol{g}(\bar{\boldsymbol{x}}_k, \bar{\boldsymbol{u}}_k))\Delta w_k|^2 + \\ & 2\langle(\boldsymbol{g}(\boldsymbol{x}_k, \boldsymbol{u}_k) - \boldsymbol{g}(\bar{\boldsymbol{x}}_k, \bar{\boldsymbol{u}}_k))\Delta w_k, \boldsymbol{x}_k - \bar{\boldsymbol{x}}_k\rangle + \\ & 2\nu|\boldsymbol{u}_k - \bar{\boldsymbol{u}}_k|^2 \end{aligned}$$

Note that $\Delta w_k \in \mathbf{R}^p$ is composed of independent identically distributed $\mathcal{N}(0, \Delta)$ entries which are independent of $\mathfrak{F}_{t_k}$. Then, taking expectations yield

$$\begin{aligned} (1+2\mu\Delta)\mathrm{E}|\boldsymbol{x}_{k+1} - \bar{\boldsymbol{x}}_{k+1}|^2 \le{}& \mathrm{E}|\boldsymbol{x}_k - \bar{\boldsymbol{x}}_k|^2 + \\ & \Delta\mathrm{E}|\boldsymbol{g}(\boldsymbol{x}_k, \boldsymbol{u}_k) - \boldsymbol{g}(\bar{\boldsymbol{x}}_k, \bar{\boldsymbol{u}}_k)|^2 + 2\nu\mathrm{E}|\boldsymbol{u} - \bar{\boldsymbol{u}}|^2_{\sup} \le \\ & (1+c\Delta)\mathrm{E}|\boldsymbol{x}_k - \bar{\boldsymbol{x}}_k|^2 + (2\nu + c\Delta)\mathrm{E}|\boldsymbol{u} - \bar{\boldsymbol{u}}|^2_{\sup} \end{aligned}$$

A similar analysis gives (22) for the split-step backward Euler method.

The inequality (23) follows because $\bar{\boldsymbol{x}}_k \equiv 0$ is a solution. □

The above Lemma leads to the following Theorem.

Theorem 4. Under Assumption 2, given any positive Δ, any two numerical solutions of the drift-implicit Euler method or the split-step backward Euler method satisfy the estimation

$$\mathrm{E}|\boldsymbol{x}_k - \bar{\boldsymbol{x}}_k|^2 \le \mathrm{E}|\boldsymbol{x}_0 - \bar{\boldsymbol{x}}_0|^2 \mathrm{e}^{-\hat{\gamma}(\Delta)k\Delta} + \frac{2\nu + c\Delta}{2\mu\Delta - c\Delta}\mathrm{E}|\boldsymbol{u} - \bar{\boldsymbol{u}}|^2_{\sup} \tag{24}$$

Moreover any solution satisfies the estimation

$$\mathrm{E}|\boldsymbol{x}_k|^2 \le \mathrm{E}|\boldsymbol{x}_0|^2 \mathrm{e}^{-\hat{\gamma}(\Delta)k\Delta} + \frac{2\nu + c\Delta}{2\mu\Delta - c\Delta}\mathrm{E}|\boldsymbol{u}|^2_{\sup} \tag{25}$$

That is, both the drift-implicit Euler and the split-step backward Euler methods possess the mean-square exp-ISS with rate constant $l = \hat{\gamma}(\Delta)$, growth constant $N = 1$ and gain $\gamma = (2\nu + c\Delta)/(2\mu\Delta - c\Delta)$, where

$$\hat{\gamma}(\Delta) := \frac{1}{\Delta}\log\left(\frac{1+2\mu\Delta}{1+c\Delta}\right) > 0$$

Also, given any $\varepsilon > 0$, there exists a postivie Δ^* such that for all $0 < \Delta \le \Delta^*$,

$$\mathrm{E}|\boldsymbol{x}_k|^2 \le \mathrm{E}|\boldsymbol{x}_0|^2 \mathrm{e}^{(-(2\mu - c) + \varepsilon)k\Delta} + \frac{2\nu + c\Delta}{2\mu\Delta - c\Delta}\mathrm{E}|\boldsymbol{u}|^2_{\sup} \tag{26}$$

for all $k \ge 0$.

Proof. From Lemma 6, we can easily obtain

$$\begin{aligned} \mathrm{E}|\boldsymbol{x}_k - \bar{\boldsymbol{x}}_k|^2 \le{}& \left(\frac{1+c\Delta}{1+2\mu\Delta}\right)^k \mathrm{E}|\boldsymbol{x}_0 - \bar{\boldsymbol{x}}_0|^2 + \\ & \frac{1 - (\frac{1+c\Delta}{1+2\mu\Delta})^k}{1 - \frac{1+c\Delta}{1+2\mu\Delta}} \frac{2\nu + c\Delta}{1+2\mu\Delta}\mathrm{E}|\boldsymbol{u} - \bar{\boldsymbol{u}}|^2_{\sup} \end{aligned} \tag{27}$$

Since $2\mu - c > 0$ implies $\hat{\gamma}(\Delta) = \frac{1}{\Delta}\log(\frac{1+2\mu\Delta}{1+c\Delta}) > 0$ and $\frac{1+c\Delta}{1+2\mu\Delta} < 1$, the inequality (24) follows directly from (27). The inequality (25) follows because $\bar{\boldsymbol{x}}_k \equiv 0$ is a solution, and (26) is a consequence of the fact that $\hat{\gamma}(\Delta) = 2\mu - c + O(\Delta)$. □

Remark 1. Summing up the statements of Theorems 3 and 4, we can see that the two types of implicit Euler methods successfully capture the mean-square exp-ISS of the SCS (1). Similar to Corollary 4.5 in [13], Theorem 4 can be applied for all positive Δ because of the implicit structure particularly in the methods. Furthermore, we also note that the two types of implicit Euler methods and the SCS (1) have the same growth constant, the same gain (as $\Delta = 1$) and the same rate constant (as $\Delta \to 0$).

4 Numerical examples

In order to verify the correctness of the above-stated theorems, two numerical examples were performed using Matlab. We utilized the Matlab routines for stochastic differential equations written by Higham D J. They are free to download from http://www.maths.strath.ac.uk/aas96106/algfiles.html.

Example 1. Consider the following one-dimensional linear stochastic control system

$$\mathrm{d}\boldsymbol{y}(t) = (-2\boldsymbol{y}(t) + \boldsymbol{u}(t))\mathrm{d}t + (\boldsymbol{y}(t) + \boldsymbol{u}(t))\mathrm{d}w(t), 0 \le t \le 16 \tag{28}$$

Clearly, the system (28) satisfies Assumption 2 with $\mu = 1.5, \nu = 0.5, c = 2$. Then, Theorem 3 tells that (28) possesses the mean-square exp-ISS with rate constant $\lambda = 1$, growth constant $M = 1$ and gain $\beta = 3$, that is, the estimation

$$\mathrm{E}|\boldsymbol{y}(t)|^2 \le \mathrm{E}|\boldsymbol{\xi}|^2 \mathrm{e}^{-t} + 3\mathrm{E}|\boldsymbol{u}|^2_{\sup} \tag{29}$$

holds. Using the random number generator "randn" in Matlab which can generate normally distributed pseudo random numbers, the stochastic θ-methods (with $\theta = 0, 0.5, 1$) are carried out through 1 000 sample paths over $0 \le t \le 16$, with the step-size $\Delta = 2^{-2}$, the initial condition $\boldsymbol{\xi} = 2$ and the input $\boldsymbol{u}(t)$ varies randomly over $[0, 1]$.

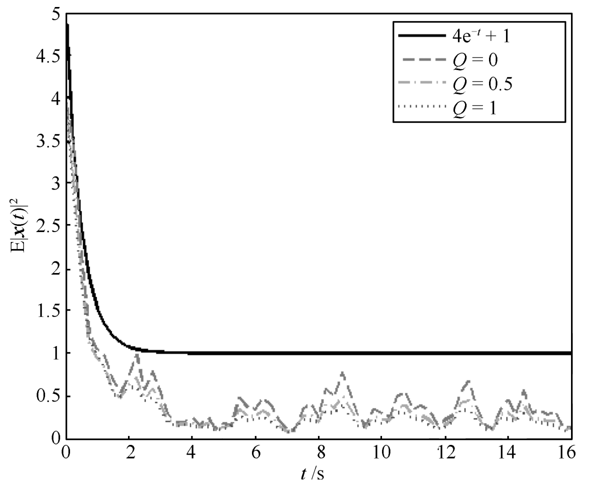

Fig. 1 $\mathrm{E}|\boldsymbol{x}(t)|^2$ of solutions of the stochastic θ-methods for (28)

From Fig. 1, we see that the solutions of stochastic θ-methods satisfy the estimation

$$\mathrm{E}|\boldsymbol{x}(t)|^2 \le \mathrm{E}|\boldsymbol{\xi}|^2 \mathrm{e}^{-t} + 1$$

for all the θ values. Thus, by comparing the inequality (29) and Fig. 1, we can say that Theorem 2 is well verified, for the stochastic θ-methods (with $\theta = 0, 0.5, 1$) share the mean-square exp-ISS with system (28).

Example 2. Consider the following one-dimensional nonlinear stochastic control system

$$\mathrm{d}\boldsymbol{y}(t) = (-2\boldsymbol{y}(t) - \boldsymbol{y}^3(t) + \boldsymbol{u}(t))\mathrm{d}t + (\boldsymbol{y}(t) + \boldsymbol{u}(t))\mathrm{d}w(t), \quad 0 \le t \le 16 \tag{30}$$

Note that $(\boldsymbol{y} - \bar{\boldsymbol{y}})(\boldsymbol{y}^3 - \bar{\boldsymbol{y}}^3) \ge 0$ for any $\boldsymbol{y}, \bar{\boldsymbol{y}} \in \mathbf{R}$. Then, we can easily show that the system (30) satisfies Assumption 2 with $\mu = 1.5, \nu = 0.5, c = 2$. Hence, Theorem 3 yields the estimation

$$\mathrm{E}|\boldsymbol{y}(t)|^2 \le \mathrm{E}|\boldsymbol{\xi}|^2 \mathrm{e}^{-t} + 3\mathrm{E}|\boldsymbol{u}|^2_{\sup} \tag{31}$$

i.e., (30) possesses the mean-square exp-ISS with rate constant $\lambda = 1$, growth constant $M = 1$ and gain $\beta = 3$.

Numerical solutions of (30) by the Euler-Maruyama, the drift-implicit Euler and the split-step backward Euler methods with 1 000 sample paths each are displayed in Fig. 2. Here we employed the step-size $\Delta = 2^{-2}$, the initial condition $\boldsymbol{\xi} = 4$ and the input $\boldsymbol{u}(t)$ varies randomly over $[0, 1]$.

From Fig. 2, we can see that the Euler-Maruyama method does not reproduce the mean-square exp-ISS of the nonlinear stochastic system (30), while the drift-implicit Euler and the split-step backward Euler methods well inherit the stability property. This confirms Theorem 4.

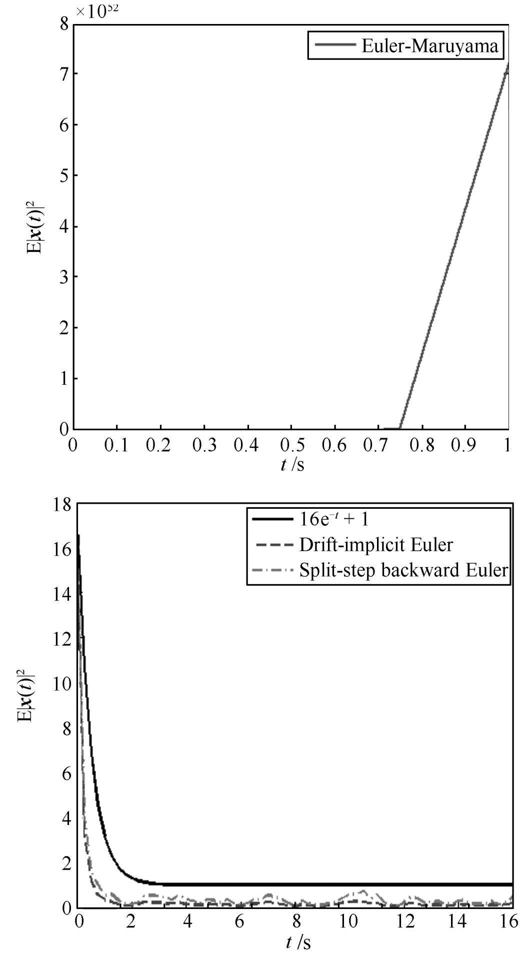

Fig. 2 $\mathrm{E}|\boldsymbol{x}(t)|^2$ of solutions of the three Euler methods for (30)

5 Conclusions

In this paper, we have restricted ourselves to the mean-square exp-ISS of stochastic θ-method and implicit Euler methods for SCSs. We have shown that the mean-square exp-ISS of an SCS holds if and only if that of the stochastic θ-method (for sufficiently small step sizes) is preserved. It is interesting and important to analyze that how long the step sizes can ensure the above result, or whether the step size can be represented by the system parameters. However, this problem is a challenge, and can be made as our next research focus. Furthermore, for a class of SCSs with a one-sided Lipschitz drift, we have also shown that two implicit Euler methods (for any step sizes) can inherit the mean-square exp-ISS property of the SCSs.

References

1 Sontag E D. Smooth stabilization implies coprime factorization. *IEEE Transactions on Automatic Control*, 1989, **34**(4): 435−443

2 Angeli D. An almost global notion of input-to-state stability. *IEEE Transactions on Automatic Control*, 2004, **49**(6): 866−874

3 Sontag E D, Wang Y. On characterizations of the input-to-state stability property. *Systems & Control Letters*, 1995, **24**(5): 351−359

4 Sontag E D, Wang Y. New characterizations of input-to-state stability. *IEEE Transactions on Automatic Control*, 1996, **41**(9): 1283−1294

5 Xie W X, Wen C Y, Li Z G. Input-to-state stabilization of switched nonlinear systems. *IEEE Transactions on Automatic Control*, 2001, **46**(7): 1111−1116

6 Yeganegar N, Pepe P, Dambrine M. Input-to-state stability of time-delay systems: a link with exponential stability. *IEEE Transactions on Automatic Control*, 2008, **53**(6): 1526−1531

7 Sun F L, Guan Z H, Zhang X H, Chen J C. Exponential-weighted input-to-state stability of hybrid impulsive switched systems. *IET Control Theory and Applications*, 2012, **6**(3): 430−436

8 Liu J, Liu X Z, Xie W C. Input-to-state stability of impulsive and switching hybrid systems with time-delay. *Automatica*, 2011, **47**(5): 899−908

9 Zhao P, Feng W, Kang Y. Stochastic input-to-state stability of switched stochastic nonlinear systems. *Automatica*, 2012, **48**(10): 2569−2576

10 Spiliotis J, Tsinias J. Notions of exponential robust stochastic stability, ISS and their Lyapunov characterizations. *International Journal of Robust and Nonlinear Control*, 2003, **13**(2): 173−187

11 Tsinias J. The concept of exponential input to state stability for stochastic systems and applications to feedback stabilization. *Systems & Control Letters*, 1999, **36**(3): 221−229

12 Tsinias J. Stochastic input-to-state stability and applications to global feedback stabilization. *International Journal of Control*, 1998, **71**(5): 907−930

13 Higham D J, Mao X R, Stuart A M. Exponential mean square stability of numerical solutions to stochastic differential equations. *LMS Journal of Computation and Mathematics*, 2003, **6**: 297−313

14 Higham D J. Mean-square and asymptotic stability of the stochastic theta method. *SIAM Journal on Numerical Analysis*, 2000, **38**(3): 753−769

15 Mao X R. Exponential stability of equidistant Euler-Maruyama approximations of stochastic differential delay equations. *Journal of Computational and Applied Mathematics*, 2007, **200**(1): 297−316

16 Saito Y, Mitsui T. Stability analysis of numerical schemes for stochastic differential equations. *SIAM Journal on Numerical Analysis*, 1996, **33**(6): 2254−2267

17 Hu G D, Liu M Z. Input-to-state stability of Runge-Kutta methods for nonlinear control systems. *Journal of Computation and Applied Mathematics*, 2007, **205**(1): 633−639

18 Zhu Q, Hu G D, Zeng L. Mean-square exponential input-to-state stability of Euler-Maruyama method applied to stochastic control systems. *Acta Automatica Sinica*, 2010, **36**(3): 406−411

19 Dekker K, Verwer J G. *Stability of Runge-Kutta Methods for Stiff Nonlinear Equations*. Amsterdam: North-Holland, 1984

20 Higham D J, Mao X R, Stuart A M. Strong convergence of Euler-type methods for nonlinear stochastic differential equations. *SIAM Journal on Numerical Analysis*, 2002, **40**(3): 1041−1063

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