Information Topology-independent Consensus Criteria for Second-order Systems under Directed Graph

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Abstract In this paper, consensus seeking of second-order systems without leaders is investigated under possibly switching directed graphs. Two consensus algorithms using different cooperative schemes are proposed and some information topology-independent criteria are obtained. For the first one, an eigenvalue-based analysis is taken to attain a sufficient and necessary condition for consensus seeking under fixed directed graph. For the second one, consensus can be achieved as long as the union of the switching graphs has a directed spanning tree frequently enough. Convergence analysis is presented, which is facilitated by an equivalent model transformation into a cascaded system. A novel sufficient and necessary condition for consensus seeking under switching undirected graph is also obtained using the same strategy. Moreover, robustness of both the algorithms to time-delays is studied under fixed directed graph. Illustrative examples are also provided to show the effectiveness of the theoretical results.

Key words Consensus, directed graph, cooperative control, multi-agent systems, time delay

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During the past decade, a surge of attention has been paid to the consensus problem of multi-agent systems for its broad application in formation control[1−2], flocking[3−5], rendezvous[6], attitude synchronization[7−9] and so on. The aim of consensus is to design a distributed control law so that an agreement can be achieved for all agents in the network. Yet, the consensus problem is not new and it has a long history in computer science and especially in the field of automatic control and distributed computation[10]. It regains large amount of interest in recent years because of the work of Reynolds[11], Vicsek et al.[12], and Jadbabaie et al.[13], where both mathematical simulations and theoretical analysis indicate the emergence of collective behaviors of a multi-agent system via local information interchange. As is proved in [1] and [14], the communication constraints of the network play key roles in the system stability and performance. Motivated by these results, numerous efforts have been made to study consensus problems under different communication conditions. To name a few, Olfati-Saber et al.[15] prove that connected undirected graph or strongly connected and balanced graph is sufficient and necessary for average consensus with fixed topology. This result is extended to consensus seeking with dynamically changing topologies for both continuous-time and discontinuous-time cases in [16], where relaxed conditions are drawn for consensus seeking. Similar result can also be found in [17]. Apart from these leaderless consensus seeking problems, consensus problems with leaders are also investigated under the framework of leader-following consensus[18], consensus tracking[19] and pinning control[20]. Also, consensus under networks with delayed communications is also studied in the literature by Lyapunov approaches or frequency domain analysis, see [21−22] for example. All the aforementioned work assumes that the communication or information interchange undergoes all the time, which cannot well describe the interactions among the agents in some real situations. To cope with this problem, intermittent information interchange is taken into consideration in [23−24], and recently in [25−26] where a novel consensus protocol using synchronous information transmission is proposed for multi-agent systems with second-order dynamics and nonlinear dynamics with/without external disturbances, respectively. For more details, please refer to the related literature.

Generally speaking, the main focus of interest in consensus problems falls into the field of systems with single- or double− integrator dynamics (see [16−17], [27−31] and references therein). It is well known that the first-order consensus can be achieved under very weak conditions even when the undirected graph is not connected or the directed graph has no directed spanning tree. For example, Ren et al.[16] prove that a directed topology having a spanning tree frequently enough can ensure the consensus of first-order systems. Similar result is also given in [17]. For networks with diverse input and communication delays, Tian et al.[22] show that the first-order consensus can still be achieved under certain condition which is independent of asymmetric communication delays as long as the information flow has a directed spanning tree. Recently, using so-called infinite integral graph, a novel result on first-order consensus under undirected switching graph is obtained in [32].

Unlike first-order consensus, consensus of second-order systems is thought to be more challenging in the literature. As is shown in [33−34], consensus may fail to be achieved even if the fixed graph has a directed spanning tree. In addition, the closed-loop system may lose its stability when more connections are added to the network. The negative point results from the fact that existing criteria for consensus achieving of second-order systems always involves the information topology of the network, i.e., it is characterized by the eigenvalues of the Laplacian matrix or the weighting factors of the communication links (see [33−34], and references therein). This means that the closed-loop system is quite sensitive to the information topology of the network and thus is prone to problems. For example, tedious calculation of eigenvalues is required to get a stable control gain for large-scale systems such as network with tens of thousands of sensors. Moreover, much more efforts have to be made to improve the robustness to the dynamically changing information topology. In addition, if the weighting factors are already determined for example by optimization, undesired dynamics such as overshooting transient response may occur because it is unable to choose control gains freely. Therefore, it does make sense to develop algorithms which can achieve consensus independently of the information topology.

In this paper, we investigate leaderless consensus of

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second-order systems with possibly switching directed graph. The content can be divided into two parts. In the first part, we develop a two-hop consensus algorithm and a sufficient and necessary condition for consensus seeking under fixed directed graph is obtained.The robustness to uniform constant delay is further discussed. In the second part, motivated by the very recent work in [35], we first investigate a variant of a consensus algorithm, which is originally proposed in [36], with a switching directed topology. It is proved that the second-order system can achieve consensus asymptotically as long as the union of the switching topology has a directed spanning tree frequently enough. Compared with a similar result drawn in [35], our consensus criterion admits of piecewise continuous weight factors from any bounded set and hence applies to all switching directed topologies but those switching infinitely fast. Note that it is a more general and practical issue and is left unresolved in [35], thus we fill the gap in this sense. In addition, we drive a sufficient and necessary condition for undirected switching graph with the notion of integral graph as a special case. To our best knowledge, no similar results on second-order consensus have been reported in the literature. Afterwards, same strategy is extended to deal with the case of system with diverse communication delay under fixed directed graph. We prove that consensus can be achieved independently of the time-delays as well as the network topology, which extends the relevant result in [37].

The main contribution of this study is to present several information topology-independent criteria for consensus seeking, which relax and improve some of the obtained results in the relevant literature. The rest of the paper is organized as follows. In Section 1, preliminaries on notions of graph and some lemmas are given. In Section 2, main results obtained in this paper are presented. Illustrative examples and conclusions are given respectively in Section 3 and Section 4.

1 Preliminaries and problem formulation

1.1 Graph

A directed graph can be described by $G=(V,E)$, where $V=\{1,2,\cdots,n\}$ is the node set, $E\subset V\times V$ is the ordered edge set. A directed path is an edge sequence of the form (i_1,i_2), (i_2,i_3), (i_3,i_4), $\cdots$. A graph is said to have a directed spanning tree if there is a node which has a directed path to all the other nodes in the graph. The union of two graphs is a graph whose node set and edge set are respectively the unions of the node sets and edge sets of the two graphs. The weighted adjacent matrix $A=[a_{ij}]\in\mathbf{R}^{n\times n}$ of a directed graph is a matrix with non-negative elements, and $a_{ij}>0$ if and only if $(j,i)\in E_n$. The Laplacian matrix of a directed graph is defined by $L=D-A$, where so-called in-degree matrix $D=[d_{ij}]$ is a diagonal matrix with the diagonal elements $d_{ii}=\sum_{j=1}^{n}a_{ij}$. The neighbor set of node i is a collection of all the nodes that has a directed edge pointing to i. If the edge set is unordered pair of node set, i.e., $(j,i)\in E_n$ implies $(i,j)\in E_n$, and the adjacent matrix A is symmetric, then G is an undirected graph. An undirected graph is connected if there is an undirected path between any two distinct nodes.

When graph G is used to describe the information flow of a multi-agent system, an edge $(i,j)\in E$ implies that agent j can receive information from agent i. Note that both the information and its adjacency matrix are not necessarily time-invariant.

1.2 Useful lemmas

Lemma 1[16]**.** Let L be the Laplacian matrix of a fixed directed graph, then L has a simple zero eigenvalue and all the other eigenvalues have positive real parts if and only if the graph has a directed spanning tree.

Lemma 2[38]**.** Let $A(t)$, $L(t)$ be the adjacency matrix and Laplacian matrix of a switching directed graph $G(t)$, respectively. Suppose that $A(t)$ is piecewise continuous and its nonzero and hence positive entries are both uniformly lower and upper bounded. Let $t_0,t_1,\cdots$ be the time at which $A(t)$ switches, and let $t_i-t_{i-1}\geq t_{\rm L}$, $\forall i=1,2,\cdots$ with $t_{\rm L}$ being a positive constant. Then, $\dot{\boldsymbol{x}}(t)=-L(t)\boldsymbol{x}(t)$ achieves consensus if there exists an infinite sequence of contiguous, nonempty, uniformly bounded time-intervals $\big[t_{i_j},t_{i_{j+1}}\big)$, $j=1,2,\cdots$ starting at $t_{i_1}=t_0$, with the property that the union of the directed graphs across each such interval has a directed spanning tree.

Lemma 3[38]**.** Let $C(t)=[c_{ij}(t)]\in\mathbf{R}^{n\times n}$ be piecewise continuous, where $c_{ij}(t)\geq 0$, $\forall i\neq j$, and $\sum_j c_{ij}=0$, and $\Phi_C(t,t_0)$ be the corresponding transition matrix. Then $\Phi_C(t,t_0)$ is a row-stochastic matrix with positive diagonal entries for any $t\geq t_0$.

Lemma 4[39−40]**.** If $\boldsymbol{u}(t)=\int_{t_0}^{t}{\rm e}^{-\alpha(t-\tau)}\boldsymbol{\omega}(\tau)\,{\rm d}\tau$, where $\boldsymbol{\omega}(t)$ is bounded and $\lim_{t\to\infty}\boldsymbol{\omega}(t)=\boldsymbol{\omega}_\infty$, then $\lim_{t\to\infty}\boldsymbol{u}(t)=\boldsymbol{\omega}_\infty/\alpha$.

Lemma 5[41]**.** Consider the exponential polynomial

$$\begin{aligned}&{\rm P}\left(\lambda,{\rm e}^{-\lambda\tau_1},\cdots,{\rm e}^{-\lambda\tau_m}\right)=\\&\quad\lambda^n+p_1^{(0)}\lambda^{n-1}+\cdots p_{n-1}^{(0)}\lambda+p_n^{(0)}+\\&\quad\left[p_1^{(1)}\lambda^{n-1}+\cdots p_{n-1}^{(1)}\lambda+p_n^{(1)}\right]{\rm e}^{-\lambda\tau_1}+\cdots+\\&\quad\left[p_1^{(m)}\lambda^{n-1}+\cdots p_{n-1}^{(m)}\lambda+p_n^{(m)}\right]{\rm e}^{-\lambda\tau_m}\end{aligned}$$

where $\tau_i\geq 0\,(i=1,2,\cdots,m)$ and $p_j^{(i)}(i=1,2,\cdots,m;\ j=1,2,\cdots,n)$ are constants. As $(\tau_1,\tau_2,\cdots,\tau_m)$ vary, the sum of the orders of the zeros of ${\rm P}\left(\lambda,{\rm e}^{-\lambda\tau_1},\cdots,{\rm e}^{-\lambda\tau_m}\right)$ on the open right-half plane can change only if a zero appears on or crosses the imaginary axis.

1.3 Problem formulation

We consider a network of n agents under a weighted directed graph described above. The agents are governed by the following double-integrator dynamics

$$\begin{aligned}&\dot{\boldsymbol{r}}_i=\boldsymbol{v}_i,\\&\dot{\boldsymbol{v}}_i=\boldsymbol{u}_i,\quad i=1,2,\cdots,n\end{aligned}\tag{1}$$

where $\boldsymbol{r}_i\in\mathbf{R}^m$ and $\boldsymbol{v}_i\in\mathbf{R}^m$ are the position and velocity of agent i, respectively, $\boldsymbol{u}_i\in\mathbf{R}^m$ is the control input. The aim of the second-order consensus is to design a distributed control law such that system (1) achieves consensus asymptotically, i.e., for all initial conditions, $\boldsymbol{r}_i(t)\to\boldsymbol{r}_j(t)$ and $\boldsymbol{v}_i(t)\to\boldsymbol{v}_j(t)$ as $t\to\infty$, $\forall i,j=1,2,\cdots,n$.

2 Consensus algorithms and information topology-independent criteria

In this section, two consensus algorithms using different cooperative schemes are designed. Criteria which are independent of the information topology are obtained under fixed directed graph and switching directed graph, respectively. The robustness of both the proposed algorithms to time-delays under fixed directed graph is also given immediately after the analysis of the basic algorithms.

2.1 Consensus algorithm under fixed topology

Provided that the information flow is fixed and time-invariant, the following second-order consensus algorithm is designed

$$\boldsymbol{u}_i = -\gamma \sum_{j=1}^{n} a_{ij} (\boldsymbol{v}_i - \boldsymbol{v}_j) - \sum_{j=1}^{n} l_{ij} \sum_{k=1}^{n} a_{jk} (\boldsymbol{r}_j - \boldsymbol{r}_k), \quad \forall i = 1, 2, \cdots, n \tag{2}$$

where $\gamma > 0$ is a positive constant, a_{ij} is the element of the weighted adjacency matrix and l_{ij} is the element of the Laplacian matrix.

Note that the second term of the right-hand side of (2) equals $-l_{ii} \sum_{k=1}^{n} a_{ik} (\boldsymbol{r}_i - \boldsymbol{r}_k)$ plus the sum of $-l_{ij} \sum_{k=1}^{n} a_{jk} (\boldsymbol{r}_j - \boldsymbol{r}_k)$ or $a_{ij} \sum_{k=1}^{n} a_{jk} (\boldsymbol{r}_j - \boldsymbol{r}_k)$ for all $j \neq i$. If agent j is a neighbor of agent i, then $a_{ij} \sum_{k=1}^{n} a_{jk} (\boldsymbol{r}_j - \boldsymbol{r}_k)$ is a collection of the relative positions between agent j and its neighbors. Also note that $l_{ii} \sum_{k=1}^{n} a_{ik} (\boldsymbol{r}_i - \boldsymbol{r}_k)$ can be seen as a weighted sum of the relative positions between agent i and its neighbors with new weight factor $l_{ii}a_{ik}$. Therefore, the consensus algorithm (2) is a valid distributed cooperative control law, and can be seen as a two-hop consensus algorithm in the sense that the positions of one′s neighbor′s neighbors are required to calculate the input.

2.1.1 Topology-independent consensus criterion

For algorithm (2), we have the following statement:

Theorem 1. If the directed graph is fixed, then consensus of system (1) using second-order consensus algorithm (2) with all $\gamma \geq 2$ is achieved if and only if the graph has a directed spanning tree. In addition, if the consensus is achieved, then $\boldsymbol{r}(t) - 1_n \boldsymbol{v}^{\mathrm{T}} \otimes I_m (\boldsymbol{r}(0) + \boldsymbol{v}(0)t) \to 0$ and $\boldsymbol{v}(t) - 1_{\mathrm{n}} \boldsymbol{v}^{\mathrm{T}} \otimes I_m \boldsymbol{v}(0) \to 0$ as $t \to \infty$, where $\boldsymbol{r}(t) = \left[\boldsymbol{r}_1^{\mathrm{T}}, \boldsymbol{r}_2^{\mathrm{T}}, \cdots, \boldsymbol{r}_n^{\mathrm{T}}\right]^{\mathrm{T}}$, $\boldsymbol{v}(t) = \left[\boldsymbol{v}_1^{\mathrm{T}}, \boldsymbol{v}_2^{\mathrm{T}}, \cdots, \boldsymbol{v}_n^{\mathrm{T}}\right]^{\mathrm{T}}$, and $\boldsymbol{v}$ is the unique left eigenvector of L associated with the zero eigenvalue satisfying $\boldsymbol{v}^{\mathrm{T}} 1_n = 1$.

Proof. Note that if the information flow has no directed spanning tree, then at least two agents can neither directly nor indirectly communicate with each other, which implies it is unable to cooperate. Hence, the necessity is obvious. In the following, we merely prove its sufficiency part.

Sufficiency. The closed-loop system of (1) and (2) can be written in the matrix form as

$$\begin{bmatrix} \dot{\boldsymbol{r}} \\ \dot{\boldsymbol{v}} \end{bmatrix} = (\Theta \otimes I_m) \begin{bmatrix} \boldsymbol{r} \\ \boldsymbol{v} \end{bmatrix} \tag{3}$$

where

$$\Theta = \begin{bmatrix} 0 & I_n \\ -L^2 & -\gamma L \end{bmatrix} \tag{4}$$

Since the directed graph is fixed, the solution of (3) can be given by

$$\begin{bmatrix} \boldsymbol{r}(t) \\ \boldsymbol{v}(t) \end{bmatrix} = \left(\mathrm{e}^{\Theta t} \otimes I_m\right) \begin{bmatrix} \boldsymbol{r}(0) \\ \boldsymbol{v}(0) \end{bmatrix} \tag{5}$$

Note that the eigenvalues of Θ can be derived from $\det(\lambda I_{2n} - \Theta) = 0$, and also note that

$$\det(\lambda I_{2n} - \Theta) = \det\left(\begin{bmatrix} \lambda I_n & -I_n \\ L^2 & \lambda I_n + \gamma L \end{bmatrix}\right) = \det\left(\lambda^2 I_n + \gamma \lambda L + L^2\right) = \prod_{i=1}^{n} \left(\lambda^2 + \gamma \lambda \mu_i + {\mu_i}^2\right) \tag{6}$$

where $\det(\lambda I_{2n} - \Theta)$ is the characteristic polynomial of Θ and μ_i $(i = 1, 2, \cdots, n)$ is the eigenvalue of L. Therefore, the eigenvalues of Θ are given by

$$\lambda_{i\pm} = \frac{-\gamma \mu_i \pm \sqrt{(\gamma^2 - 4)\mu_i^2}}{2}, \ i = 1, 2, \cdots, n \tag{7}$$

Since $\gamma \geq 2$, (7) can be further written as

$$\lambda_{i\pm} = \frac{-\left(\gamma \mp \sqrt{(\gamma^2 - 4)}\right)\mu_i}{2}, \ i = 1, 2, \cdots, n \tag{8}$$

Thus, $\lambda_{i\pm} = 0$ if and only if $\mu_i = 0$. In addition, the signs of the real part of $\lambda_{i\pm}$ and the real part of $-\mu_i$ are exactly the same for each $i \in V$. Since the information flow has a directed spanning tree, L has a simple zero eigenvalue and all the others have negative real parts according to Lemma 1, which in turn implies Θ has a zero eigenvalue with algebraic multiplicity 2 and all the other eigenvalues have negative real parts.

Now we are going to show that its geometric multiplicity of zero eigenvalue equals one. To see this, let $\boldsymbol{q} = \left[\boldsymbol{q}_a^{\mathrm{T}} \quad \boldsymbol{q}_b^{\mathrm{T}}\right]^{\mathrm{T}}$ be a well-defined eigenvector of Θ associated with zero eigenvalue, i.e.,

$$\Theta \boldsymbol{q} = \begin{bmatrix} 0 & I_n \\ -L^2 & -\gamma L \end{bmatrix} \begin{bmatrix} \boldsymbol{q}_a \\ \boldsymbol{q}_b \end{bmatrix} = 0 \tag{9}$$

Then $\boldsymbol{q}_b = 0$ and $L^2 \boldsymbol{q}_a = 0$. Since the algebraic multiplicity of zero eigenvalue of L^2 equals that of L, zero is a simple eigenvalue of L^2. Therefore, the geometric multiplicity of the zero eigenvalue of Θ equals one and $\left[0^{\mathrm{T}} \quad {1_n}^{\mathrm{T}}\right]^{\mathrm{T}}$ is an associated eigenvector.

Write Θ in its Jordan canonical form as

$$\Theta = PJP^{-1} = [\boldsymbol{p}_1 \quad \boldsymbol{p}_2 \quad \cdots \quad \boldsymbol{p}_{2n}] \times \begin{bmatrix} 0 & 1 & 0_{1\times(2n-2)} \\ 0 & 0 & 0_{1\times(2n-2)} \\ 0_{(2n-2)\times 1} & 0_{(2n-2)\times 1} & J' \end{bmatrix} \begin{bmatrix} \boldsymbol{v}_1^{\mathrm{T}} \\ \boldsymbol{v}_2^{\mathrm{T}} \\ \vdots \\ \boldsymbol{v}_{2n}^{\mathrm{T}} \end{bmatrix} \tag{10}$$

where $\boldsymbol{p}_i \in \mathbf{R}^{2n} (i = 1, 2, \cdots, n)$ is the right eigenvector or generalized right eigenvector of Θ, $\boldsymbol{v}_i \in \mathbf{R}^{2n} (i = 1, 2, \cdots, n)$ is the left eigenvector or generalized left eigenvector of Θ, and J' is the Jordan upper diagonal block matrix associated with the nonzero eigenvalues.

Without loss of generality, we can choose a right eigenvector $\boldsymbol{p}_1 = \left[{1_n}^{\mathrm{T}} \quad 0^{\mathrm{T}}\right]^{\mathrm{T}}$ and a generalized right eigenvector $\boldsymbol{p}_2 = \left[0^{\mathrm{T}} \quad {1_n}^{\mathrm{T}}\right]^{\mathrm{T}}$. Then it can be verified that $\boldsymbol{v}_1 = \left[\boldsymbol{v}^{\mathrm{T}} \quad 0^{\mathrm{T}}\right]^{\mathrm{T}}$ and $\boldsymbol{v}_2 = \left[0^{\mathrm{T}} \quad \boldsymbol{v}^{\mathrm{T}}\right]^{\mathrm{T}}$ are the generalized left eigenvector and the left eigenvector, where $\boldsymbol{v}$ is the nonnegative left eigenvector of L satisfying $\boldsymbol{v}^{\mathrm{T}} 1_n = 1$. Then, we obtain

$$\mathrm{e}^{\Theta t} = \mathrm{e}^{PJP^{-1}} = P \begin{bmatrix} 1 & t & 0_{1\times(2n-2)} \\ 0 & 1 & 0_{1\times(2n-2)} \\ 0_{(2n-2)\times 1} & 0_{(2n-2)\times 1} & \mathrm{e}^{J'} \end{bmatrix} P^{-1} = \boldsymbol{p}_1 \boldsymbol{v}_1^{\mathrm{T}} + (\boldsymbol{p}_1 t + \boldsymbol{p}_2) \boldsymbol{v}_2^{\mathrm{T}} + Q = \begin{bmatrix} 1_n \boldsymbol{v}^{\mathrm{T}} & 1_n \boldsymbol{v}^{\mathrm{T}} t \\ 0 & 1_n \boldsymbol{v}^{\mathrm{T}} \end{bmatrix} + Q \tag{11}$$

where $Q = [\boldsymbol{p}_3 \quad \boldsymbol{p}_4 \quad \cdots \quad \boldsymbol{p}_{2n}] \, \mathrm{e}^{J'} [\boldsymbol{v}_3 \quad \boldsymbol{v}_4 \quad \cdots \quad \boldsymbol{v}_{2n}]^{\mathrm{T}}$.

Since $Q \to 0$ as $t \to \infty$, it follows from (9) and (5) that $\boldsymbol{r}(t) - 1_n \boldsymbol{v}^{\mathrm{T}} \otimes I_m (\boldsymbol{r}(0) + \boldsymbol{v}(0)t) \to 0$ and $\boldsymbol{v}(t) - 1_{\mathrm{n}} \boldsymbol{v}^{\mathrm{T}} \otimes I_m \boldsymbol{v}(0) \to 0$ as $t \to \infty$. □

Remark 1. When the directed graph changes over time, a similar conclusion like Theorem 5.2 in [33] can be obtained. It can be verified that consensus can still be achieved with no more efforts but to ensure that the graph has a directed spanning tree at all time-intervals and the dwell time is large enough.

2.1.2 Robustness to time-delays

Consider a network with input delays. If the time-delays are constant and uniform, then the delay-free algorithm can be modified as follows:

$$\begin{aligned}\boldsymbol{u}_i = &-\gamma\sum_{j=1}^{n} a_{ij}(\boldsymbol{v}_i(t-\tau)-\boldsymbol{v}_j(t-\tau))-\\ &\sum_{j=1}^{n} l_{ij}\sum_{k=1}^{n} a_{jk}(\boldsymbol{r}_j(t-\tau)-\boldsymbol{r}_k(t-\tau))\end{aligned}\tag{12}$$

where $\tau > 0$ is the constant time-delay and $\gamma > 0$.

The closed-loop system in this case can be written in the matrix form

$$\begin{aligned}\dot{\boldsymbol{r}} &= \boldsymbol{v}\\ \dot{\boldsymbol{v}} &= -2L\boldsymbol{v}(t-\tau)-L^2\boldsymbol{r}(t-2\tau)\end{aligned}\tag{13}$$

Motivated by Theorem 2 in [34], we have the following results:

Lemma 6. System (13) has a zero eigenvalue, and its algebraic multiplicity equals $2k$ if and only if the algebraic multiplicity of the zero eigenvalue of L equals k.

Proof. The characteristic equation of (13) can be given by

$$\begin{aligned}&\det\left(\begin{bmatrix}\lambda I_n & 0\\ 0 & \lambda I_n\end{bmatrix}-\begin{bmatrix}0 & I_n\\ 0 & 0\end{bmatrix}-\begin{bmatrix}0 & 0\\ -L^2\mathrm{e}^{-\lambda\tau} & -\gamma L\mathrm{e}^{-\lambda\tau}\end{bmatrix}\right)=\\ &\quad\det\left(\begin{bmatrix}\lambda I_n & -I_n\\ L^2\mathrm{e}^{-\lambda\tau} & \lambda I_n+\gamma L\mathrm{e}^{-\lambda\tau}\end{bmatrix}\right)=\\ &\quad\det\left(\lambda^2 I_n+\gamma\lambda L\mathrm{e}^{-\lambda\tau}+L^2\mathrm{e}^{-\lambda\tau}\right)=\\ &\quad\prod_{i=1}^{n}\left(\lambda^2+(\gamma\lambda+\mu_i)\mu_i\mathrm{e}^{-\lambda\tau}\right)=0\end{aligned}\tag{14}$$

where μ_i $(i=1,2,\cdots,n)$ is the eigenvalue of L.

Letting $\mu=\alpha+\mathrm{j}\beta$, $\lambda=c+\mathrm{j}d$, and taking into account of $\lambda^2+(\gamma\lambda+\mu)\mu\mathrm{e}^{-\lambda\tau}=0$, we obtain

$$\begin{aligned}&c^2-d^2+\mathrm{e}^{-c\tau}((\gamma c\alpha+\alpha^2-\gamma d\beta-\beta^2)\cos(d\tau)+\\ &\quad(\gamma c\beta+\gamma d\alpha+2\alpha\beta)\sin(d\tau))=0\end{aligned}\tag{15}$$

$$\begin{aligned}&2cd+\mathrm{e}^{-c\tau}(-(\gamma c\alpha+\alpha^2-\gamma d\beta-\beta^2)\sin(d\tau)+\\ &\quad(\gamma c\beta+\gamma d\alpha+2\alpha\beta)\cos(d\tau))=0\end{aligned}\tag{16}$$

From (15) and (16), it can be verified that $c=d=0$ if $\alpha=\beta=0$ and $\alpha=\beta=0$ if $c=d=0$. Hence, $\lambda=0$ if and only if $\mu=0$. Since L has at least one zero eigenvalue, zero is surely an eigenvalue of (13). The relation of the algebraic multiplicity can easily be drawn from (14). □

Lemma 7. If the graph has a directed spanning tree, then the closed-loop system (13) has a purely imaginary root only if

$$\tau\in\Psi=\left\{\frac{\varphi_0+2k\pi}{|\omega_0|}\mid k=0,1,2,\cdots\right\}\tag{17}$$

where $\cos(\varphi_0)=\left(\alpha^2-(\gamma|\omega_0|+\beta)\beta\right)/\omega_0^2$, $\sin(\varphi_0)=(2\alpha\beta+\gamma|\omega_0|\alpha)/\omega_0^2$ with $0\le\varphi_0<2\pi$, and ω_0 is the real root of the following equation

$$\omega_0^4=\left(\alpha^2+\beta^2\right)\left(\alpha^2+(\beta+\gamma\omega_0)^2\right)\tag{18}$$

Proof. Suppose $\lambda=\mathrm{j}\omega$ is a purely imaginary root of (13). Let $\cos(\varphi)=\left(\alpha^2-(\gamma\omega+\beta)\beta\right)/\omega^2$, $\sin(\varphi)=(2\alpha\beta+\gamma\omega\alpha)/\omega^2$ with $0\le\varphi<2\pi$, it follows from (15) and (16) that

$$-\omega^2+\sqrt{(\alpha^2+\beta^2)\left(\alpha^2+(\beta+\gamma\omega)^2\right)}\cos(\varphi-\omega\tau)=0\tag{19}$$

$$\sin(\varphi-\omega\tau)=0\tag{20}$$

Therefore, $\varphi-\omega\tau=2k\pi$ $(k=0,\pm1,\pm2,\cdots)$ and $\omega^2=\sqrt{(\alpha^2+\beta^2)\left(\alpha^2+(\beta+\gamma\omega)^2\right)}$. Using the fact that $\tau>0$, we know $\tau=(\varphi-2k\pi)/\omega$ $(k=0,-1,-2,\cdots)$ if $\omega>0$ and $\tau=(2k\pi-\varphi)/|\omega|$ $(k=1,2,\cdots)$ if $\omega<0$.

Note that if $\mu=\alpha+\mathrm{j}\beta$ is an eigenvalue of L, then $\mu=\alpha-\mathrm{i}\beta$ is also an eigenvalue of L. Also note that if ω is a real root of (18) for some β, then $-\omega$ is a real root of (18) for $-\beta$ as well. Hence, for each φ given above, there exists a φ' such that $\cos\left(\varphi'\right)=\left(\alpha^2-(\gamma(-\omega)+(-\beta))(-\beta)\right)/(-\omega)^2=\cos(\varphi)$ and $\sin\left(\varphi'\right)=(2\alpha(-\beta)+\gamma(-\omega)\alpha)/(-\omega)^2=-\sin(\varphi)$, which means that φ' can be given by $\varphi'=2\pi-\varphi$. Therefore, we have

$$\begin{aligned}\tau\in\Psi=&\left\{\frac{\varphi-2k\pi}{\omega}\mid\omega>0,k=0,-1,-2,\cdots\right\}\bigcup\\ &\left\{\frac{2k\pi-\varphi}{|\omega|}|\omega<0,\ k=1,2,\cdots\right\}=\\ &\left\{\frac{\varphi+2k\pi}{|\omega|}\mid\omega>0,k=0,1,2,\cdots\right\}\bigcup\\ &\left\{\frac{2\pi-\varphi+2k\pi}{|\omega|}\mid\omega<0,k=0,1,2,\cdots\right\}=\\ &\left\{\frac{\varphi+2k\pi}{|\omega|}\mid\omega>0,k=0,1,2,\cdots\right\}\bigcup\\ &\left\{\frac{\varphi'+2k\pi}{|\omega|}\mid\omega<0,k=0,1,2,\cdots\right\}=\\ &\left\{\frac{\varphi_0+2k\pi}{|\omega_0|}\mid k=0,1,2,\cdots\right\}\end{aligned}\tag{21}$$

□

Theorem 2. Suppose the directed graph is fixed and $\gamma\ge2$. The retarded consensus algorithm (12) can achieve consensus if the graph has a directed spanning tree and

$$\tau<\tau_{\max}=\min\left\{\frac{\varphi_{0i}}{|\omega_{0i}|}\right\}\tag{22}$$

where $\cos(\varphi_{0i})=\left(\alpha_i^2-(\gamma|\omega_{0i}|+\beta_i)\beta_i\right)/\omega_{0i}^2$, $\sin(\varphi_{0i})=((2\beta_i+\gamma|\omega_{0i}|)\alpha_i)/\omega_{0i}^2$ with $0\le\varphi_{0i}<2\pi$, and ω_{0i} is the real root of the following equation

$$\omega_{0i}^4=\left(\alpha_i^2+\beta_i^2\right)\left(\alpha_i^2+(\beta_i+\gamma\omega_{0i})^2\right)\tag{23}$$

where α_i and β_i are respectively, the real part and imaginary part of the nonzero eigenvalue of L.

Proof. From the proof of Theorem 1, we know that the second-order consensus is achieved for the delay-free case ($\tau=0$). In addition, the closed-loop system has a zero eigenvalue with algebraic multiplicity 2 and all the other eigenvalues have negative real parts. According to Lemmas 5 and 7, the signs will not change until τ reaches the smallest positive number $\tau_{\max}$ of the set $\Psi_i=\{(\varphi_{0i}+2k\pi)/|\omega_{0i}|\mid k=0,1,2,\cdots\}$. Then consensus can be obtained for all $0\le\tau<\tau_{\max}$. □

Remark 2. From (22), it follows that $0 \in \Psi_i$ and hence $\tau_{\max} = 0$ if $\varphi_{0i} = 0$, which means no time-delay is allowed in this case. However, it will never appear since $\varphi_{0i} \neq 0$ for all nonzero eigenvalues if the network has a directed spanning tree and $\gamma \geq 2$. To see this, suppose that on the contrary, $\varphi_{0i} = 0$, we have $\sin(\varphi_{0i}) = 0$ and hence $2\beta_i + \gamma|\omega_{0i}| = 0$ since $\alpha_i > 0$. Together with (23), it is obtained $\omega_{0i}^2 \geq \alpha_i^2 + \beta_i^2$. Hence, $\omega_{0i}^2 = 4\beta_i^2/\gamma^2 \geq \alpha_i^2 + \beta_i^2$, which contradicts with $\gamma \geq 2$. Therefore, Theorem 2 justifies the robustness of the two-hop consensus algorithm to uniform delay in the network. However, the consensus criterion given in (22) is still involved with the eigenvalues of the Laplacian matrix and hence is unexpected as discussed previously. In the following, we will further our efforts to find a more powerful cooperative control law so that the consensus criterion for time-delay cases is topology-independent.

2.2 Consensus algorithm under possibly switching topology

In this subsection, we investigate the consensus problem in possibly switching directed graph and hence $G(t)$, $A(t)$, and $L(t)$ are used to describe the possible switching of the topology. The algorithm to be studied is given by

$$\boldsymbol{u}_i = -\gamma\boldsymbol{v}_i - \gamma\sum_{j=1}^{n} a_{ij}(t)(\boldsymbol{r}_i - \boldsymbol{r}_j) - \sum_{j=1}^{n} a_{ij}(t)(\boldsymbol{v}_i - \boldsymbol{v}_j) \tag{24}$$

where $\gamma > 0$ is a positive constant, $a_{ij}(t) \in [\underline{a}, \bar{a}]$ with $\bar{a} > \underline{a} > 0$ if $(j, i) \in E$ and $a_{ij}(t) \equiv 0$ otherwise.

Compared with the proposed two-hop algorithms, (24) implies each agent must use $-\gamma\boldsymbol{v}_i$ as well as its relative states with its neighbors to coordinate the whole group. Hence, consensus is achieved at the cost of absolute measurements as well as relative measurements.

2.2.1 Topology-independent consensus criterion

If the directed graph is fixed, the eigenvalue-based analysis used in the previous subsection could also be taken to show the convergence of the algorithm. However, a novel approach is utilized here in order to facilitate the analysis and to show that consensus can achieve even under a switching graph that may not have a directed spanning tree.

Theorem 3. Let $t_0, t_1, \cdots$ be the time at which $A(t)$ switches and assume that $t_i - t_{i-1} \geq t_{\mathrm{L}}$, $\forall i = 1, 2, \cdots$ with t_{L} being a positive constant. Then the system of (1) and (24) achieves consensus asymptotically if there exists an infinite sequence of contiguous, nonempty, uniformly bounded time-intervals $[t_{i_j}, t_{i_{j+1}})$, $j = 1, 2, \cdots$ starting at $t_{i_1} = t_0$, with the property that the union of the directed graphs across each such interval has a directed spanning tree.

Proof. Write (1) and (24) in matrix form:

$$\begin{aligned} \dot{\boldsymbol{r}} &= \boldsymbol{v} \\ \dot{\boldsymbol{v}} &= -\gamma(L(t) \otimes I_m)\boldsymbol{r} - (L(t) + \gamma I_n) \otimes I_m\boldsymbol{v} \end{aligned} \tag{25}$$

It is easy to be verified, with the equivalent transformation given in (27), that system (25) is equivalent to the following system:

$$\begin{aligned} \dot{\boldsymbol{x}} &= -\gamma\boldsymbol{x} + \gamma\boldsymbol{y} \\ \dot{\boldsymbol{y}} &= -(L(t) \otimes I_m)\boldsymbol{y} \end{aligned} \tag{26}$$

$$\begin{aligned} \boldsymbol{x} &= \boldsymbol{r} \\ \boldsymbol{y} &= \boldsymbol{r} + \frac{1}{\gamma}\boldsymbol{v} \end{aligned} \tag{27}$$

According to Lemma 2, the y-subsystem of (26) achieves consensus for all initial states, thus $\boldsymbol{y}(t) \to 1_n\boldsymbol{\xi}^{\mathrm{T}} \otimes I_m\boldsymbol{y}(0)$ as $t \to \infty$ for some column vector $\boldsymbol{\xi} \in \mathbf{R}^n$. Note that $\boldsymbol{y}(t)$ is given by $\boldsymbol{y}(t) = (\Phi_{L(t)}(t, t_0) \otimes I_m)\boldsymbol{y}(0)$, with $\Phi_{L(t)}(t, t_0) \otimes I_m$ being the corresponding transition matrix. Also note that $\Phi_{L(t)}(t, t_0)$ is a row-stochastic matrix according to Lemma 3. Hence, we have $\|\boldsymbol{y}(t)\|_\infty = \|(\Phi_{L(t)}(t, t_0) \otimes I_m)\boldsymbol{y}(0)\|_\infty \leq \|\boldsymbol{y}(0)\|_\infty$, i.e., $\boldsymbol{y}(t)$ is bounded by the initial states. Since (26) is a cascaded system of an exponentially stable system and a first-order consensus algorithm, with the latter being the driving system, it follows from Lemma 4 that $\boldsymbol{x}(t) \to 1_n\boldsymbol{\xi}^{\mathrm{T}} \otimes I_m\boldsymbol{y}(0)$ as $t \to \infty$. Then, using (27), it can be verified that $\boldsymbol{r}(t) \to 1_n\boldsymbol{\xi}^{\mathrm{T}} \otimes I_m(\boldsymbol{r}(0) + \boldsymbol{v}(0)/\gamma)$ and $\boldsymbol{v}(t) \to 0$ as $t \to \infty$, which means consensus of the original system (25) is achieved. □

Remark 3. The variants of consensus algorithm (24) are originally proposed in [36] for formation control under fixed topology, and are also investigated recently in [35] under both fixed and switching topology. However, the weight factors of (24) in this study could be piecewise continuous, which means the weight factors and the dwell time of the topology are no longer restricted to a finite set. Therefore, we not only extend the result of [36] to the case with switching directed topology, but also extend the result of [35] to the case with topology that could be any graph but those switching infinitely fast. In this sense, we address the general and practical issue left unresolved in [35].

The following corollary results directly from the fact that having a spanning tree is a necessity for leaderless consensus under a fixed directed graph.

Corollary 1. Assuming $A(t)$ is constant and hence $G(t)$ is fixed, consensus of (1) and (24) is achieved if and only if $G(t)$ has a directed spanning tree. In addition, if the consensus is achieved, then $\boldsymbol{r}(t) \to 1_n\boldsymbol{v}^{\mathrm{T}} \otimes I_m(\boldsymbol{r}(0) + \boldsymbol{v}(0)/\gamma)$ and $\boldsymbol{v}(t) \to 0$ as $t \to \infty$, where $\boldsymbol{v}$ is the unique left eigenvector of L associated with the zero eigenvalue satisfying $\boldsymbol{v}^{\mathrm{T}}1_n = 1$.

Remark 4. If the interactions between all the agents are bidirectional, another novel and generalized result could be drawn by taking the advantage of the results in [32]. In fact, we have the following results.

Theorem 4. If $G(t)$ is a switching undirected graph and the Laplacian matrix is piecewise continuous, then consensus is achieved if and only if its integral graph of $G(t)$ on $[0, +\infty)$ is connected (see [32] for the definition of average consensus and integral graph). In addition, if consensus is achieved, then $\boldsymbol{r}(t) \to (1/n)1_n{}^{\mathrm{T}} \otimes I_m(\boldsymbol{r}(0) + \boldsymbol{v}(0)/\gamma)$ and $\boldsymbol{v}(t) \to 0$ as $t \to \infty$.

Proof. The proof depends heavily on the fact that system $\dot{\boldsymbol{y}} = -(L(t) \otimes I_m)\boldsymbol{y}$ achieve average consensus under a switching undirected graph if and only if $G_{[0,\infty)}$ is connected, where $G_{[0,\infty)}$ is the integral graph of $G(t)$ on $[0, +\infty)$(See Theorem 4.1 in [32]). With this knowledge, we have:

Necessity. If $G_{[0,\infty)}$ is not connected, then $\boldsymbol{y}$ cannot achieve consensus, which in turn implies that $\boldsymbol{r}(t)$ and $\boldsymbol{v}(t)$ cannot achieve consensus from (27).

Sufficiency. If $G_{[0,\infty)}$ is connected, then $\boldsymbol{y}$ achieves average consensus and hence $\boldsymbol{y}(t) \to 1_n\boldsymbol{\xi}^{\mathrm{T}} \otimes I_m\boldsymbol{y}(0)$ as $t \to \infty$ with $\boldsymbol{\xi} = (1/n)1_n$ in this case. From [32], we know that $\|\boldsymbol{y}(t)\|$ is nonincreasing and every entry is bounded by $\|\boldsymbol{y}(0)\|$. Thus $\|\boldsymbol{y}(t)\|$ is bounded all the time. The rest of the proof is similar to the related part in Theorem 3 and hence is omitted. □

2.2.2 Robustness to time-delays

Considering a network with constant communication delays, we still assume the information flow is time-invariant in this case. The second-order consensus algorithm (24) now is modified to

$$\boldsymbol{u}_i = -\gamma \boldsymbol{v}_i - \gamma \sum_{j=1}^{n} a_{ij} \left(\boldsymbol{r}_i(t) - \boldsymbol{r}_j(t - \tau_{ij})) - \sum_{j=1}^{n} a_{ij} (\boldsymbol{v}_i(t) - \boldsymbol{v}_j(t - \tau_{ij}) \right) \tag{28}$$

where τ_{ij} is the constant communication delay between agents i and j.

Since τ_{ij} need not be equal to τ_{ji}, asymmetric communication delays are allowed. Before the statement on the robustness of (28), we need the following lemma which can be found in [22].

Lemma 8[22]. Considering the following continuous-time system with communication delays

$$\dot{\boldsymbol{x}}_i(t) = -\sum_{j=1}^{n} a_{ij} (\boldsymbol{x}_i(t) - \boldsymbol{x}_j(t - \tau_{ij})),\ \forall i = 1, 2, \cdots, n \tag{29}$$

consensus is achieved if the directed graph is fixed and has a directed spanning tree.

Theorem 5. Assuming $A(t)$ is constant and hence $G(t)$ is fixed, consensus of system (1) and (28) can be achieved if and only if $G(t)$ has a directed spanning tree.

Proof. It is easy to be verified that the close-loop system (1) with (28) is equivalent to the following system using the equivalent transformation (27)

$$\begin{aligned} \dot{\boldsymbol{x}}_i &= -\gamma \boldsymbol{x}_i + \gamma \boldsymbol{y}_i \\ \dot{\boldsymbol{y}}_i &= -\sum_{j=1}^{n} a_{ij} \left(\boldsymbol{y}_i(t) - \boldsymbol{y}_j(t - \tau_{ij}) \right),\ \forall i = 1, 2, \cdots n \end{aligned} \tag{30}$$

Thus, if $G(t)$ has a directed spanning tree, $\boldsymbol{y}_i$ $(i = 1, 2, \cdots, n)$ will converge to a common value according to Lemma 8, which means the second subsystem of (30) achieves consensus asymptotically. Since $\boldsymbol{y}_i(t)$ is continuously differentiable and converges as t tends to infinity, $\boldsymbol{y}_i(t)$ and hence $\boldsymbol{y}(t)$ is bounded with respect to time. The rest of the proof for the consensus of the whole system (30) proceeds along the same lines of Theorem 3 and hence is omitted again.

Note that having a directed spanning tree is necessary for reaching consensus in a fixed topology, the necessity is obvious. □

Remark 5. Leaderless consensus of second-order systems with both input delays and asymmetric communication delays is studied in [37], and a sufficient condition for consensus is given. Although the consensus criteria are independent of the communication delays, the velocity damping coefficient has to be above some constant which is weight-dependent even the input delays are equal to zero. In this sense, the consensus criterion obtained here extends the result of [37] to a more generalized condition.

Remark 6. Note that the main idea behind the analysis of Theorem 3 to Theorem 5 is to transform the second-order system to an exponentially stable system cascaded by a first-order consensus algorithm so that existing results on first-order consensus can be taken to facilitate the analysis. Thus, the results obtained here can also be extended to dynamically changing topologies with coupling delays by noting the result in [42].

3 Illustrative examples

In this section, simulations are carried out to validate the effectiveness of the proposed algorithms. For simplicity, all the results are obtained under a network with four agents whose initial states are given as $\boldsymbol{r}_1(0) = [9\quad 3\quad -7]^{\mathrm{T}}$m, $\boldsymbol{r}_2(0) = [-5\quad -2\quad 4]^{\mathrm{T}}$m, $\boldsymbol{r}_3(0) = [3\quad -6\quad 2]^{\mathrm{T}}$m, $\boldsymbol{r}_4(0) = [2\quad -3\quad 8]^{\mathrm{T}}$m, $\boldsymbol{v}_1(0) = [0.1\quad 0.2\quad -0.1]^{\mathrm{T}}$m/s, $\boldsymbol{v}_2(0) = [-0.2\quad -0.1\quad 0.3]^{\mathrm{T}}$m/s, $\boldsymbol{v}_3(0) = [-0.3\quad 0.3\quad 0.1]^{\mathrm{T}}$m/s, $\boldsymbol{v}_4(0) = [0.1\quad -0.2\quad -0.1]^{\mathrm{T}}$m/s. The weight factors of the topology are given as $a_{ij} = 1$ if $(j, i) \in E_n$ and $a_{ij} = 0$ if $(j, i) \notin E_n$.

As for the two-hop consensus algorithms (2) and (12), the network topology shown in Fig. 1 (a) is chosen. It can be verified that the graph has a directed spanning tree and that the time-delay bound given in Theorem 2 is $\tau_{\max} = 0.1895$. We chose $\gamma = 3$ and $\tau = 0.189$ so that all the assumptions in Theorem 1 and Theorem 2 are satisfied. Fig. 2 and Fig. 3 are the results of the network without delays and with delays, respectively. It is easy to see that all the positions as well as the velocities of the agents achieve consensus asymptotically in both the cases. In addition, the positions change over time while the velocities finally converge to a constant. This may be understood intuitively since only relative measurements are used to coordinate and hence we cannot expect a final convergence of the absolute states.

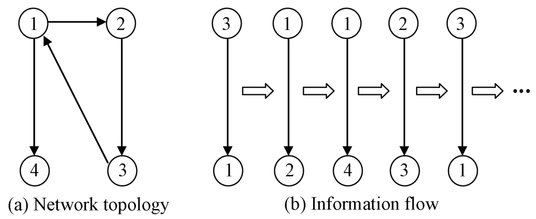

Fig. 1 Network topology and information flow

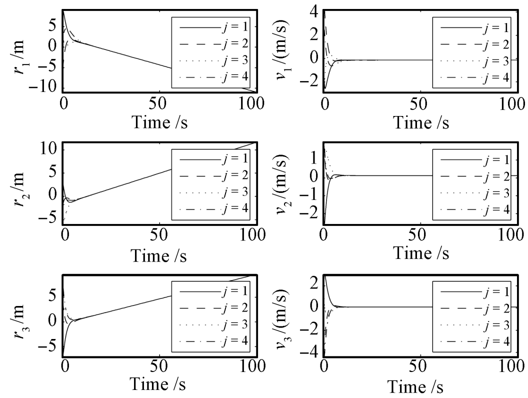

Fig. 2 Consensus of two-hop algorithm without delays

As for the one-hop consensus algorithm (24), we assume the network topology changes following the logic shown in Fig. 1 (b), where G_i $(i = 1, 2, \cdots, 5)$ is an optional topology at certain time. Specifically, the topology switches from G_i to G_{i+1} if $1 \leq i < 4$ and to G_1 otherwise, with a dwell time being is randomly chosen between 1 s and 1.5 s. Each of the weight factors is chosen to be $1 + 0.5\sin(2t)$ if it is not zero. Note that the topology union is the same as the topology given in Fig. 1 (a) which has a directed spanning tree. Thus the assumptions of Theorem 3 are satisfied. Fig. 4 presents the result of the positions and the velocities with $\gamma = 5$. To show the robustness of the algorithm against asymmetric delays, the topology is fixed and is assumed to be the union graph of the topology or Fig. 1 (a). Each of the time delays in (28) now is randomly chosen between 1.4 s and 1.6 s. The results are shown in Fig. 5. It is easy to see that both the positions and the velocities synchronize finally and hence consensus is achieved.

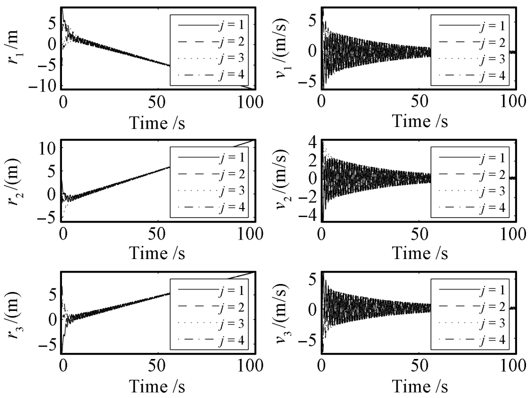

Fig. 3 Consensus of two-hop algorithm with delays

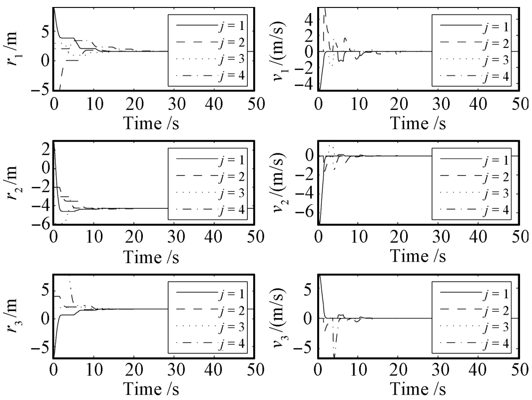

Fig. 4 Consensus of one-hop algorithm under switching topology

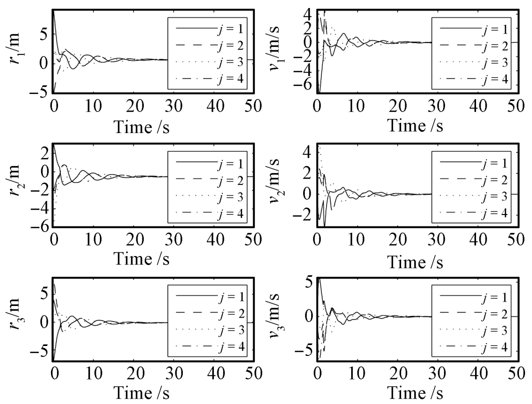

Fig. 5 Consensus of one-hop algorithm with asymmetric delays under fixed topology

4 Conclusion

Two basic leaderless consensus algorithms for second-order system with general directed graphs are proposed and both are proved to be effective under conditions which are independent of the information topology. The first one uses merely relative information to coordinate the collections and is robust to constant input delays. The main shortcoming of this part is the use of the two-hop information and the assumptions of uniform time-delays. On the other hand, the second one uses nearest-neighbor′s information to coordinate the multi-agent system. Several weak consensus criteria for first-order consensus are re-obtained for second-order consensus, which extends some results obtained in the relevant literature for both the case with and without asymmetric communication delays.

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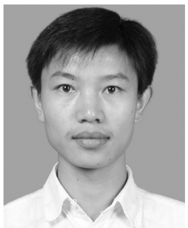

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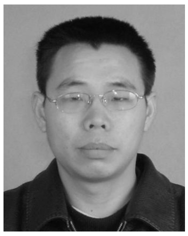

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