Robust Control Synthesis of Polynomial Nonlinear Systems Using Sum of Squares Technique

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Abstract In this paper, sum of squares (SOS) technique is used to analyze the robust state feedback synthesis problem for a class of uncertain affine nonlinear systems with polynomial vector fields. Sufficient conditions are given to obtain the solutions to the above control problem either without or with guaranteed cost or H_∞ performance objectives. Moreover, such solvable conditions can be formulated as SOS programming problems in terms of state dependent linear matrix inequalities (LMIs) which can be dealt with by the SOS technique directly. Besides, an idea is provided to describe the inverse of polynomial or even rational matrices by introducing some extra polynomials. A numerical example is presented to illustrate the effectiveness of the approach.

Key words Nonlinear, robust control, polynomial nonlinear systems, sum of squares (SOS)

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In the last few decades, the stability and performance analysis of nonlinear systems have been receiving considerable attention. Nonlinear control problems such as stability or robust stability with and without H_∞ performance[1−8], optimal control[9−11], guaranteed cost control[12] have been studied by several researchers. Despite the existed powerful theories to cope with these problems, most of the present work characterizes these control problems in terms of Hamilton Jacobi equations/inequalities (HJEs/HJIs). Unfortunately, there is no universal methodology for effectively solving the HJEs or HJIs yet. Until now, the computation issue is still one of the major concerns in nonlinear system control theory.

Sum of squares (SOS) programming, which is a method to handle non-convex polynomial programming globally by using SOS polynomials and semi-definite programming (SDP), has been used to provide tractable relaxations for control problems of nonlinear systems. So far, many applications have been found in the literature. In [13−16], by using SOS technique, the problems of state feedback H_∞ control, output feedback H_∞ control, L_2-gain computation and simultaneous stabilization of polynomial nonlinear systems were addressed, and the solutions to these problems were obtained by solving some convex state-dependent linear matrix inequalities (LMIs) that are easier to solve than HJIs or HJEs. Specifically, in [14], a new idea of introducing the exact variation rate of the state to construct the Lyapunov function potentially leads to a less conservative result than in the earlier work. In [17], a stability criterion based on the density function, which can be viewed as a dual to Lyapunov′s second theorem, was tested by using the SOS programming method. Apart from the applications mentioned above, SOS technique was used to study other control problems as well, such as generalized version of absolute stability, observer design of polynomial fuzzy systems, and stability analysis of switched nonlinear systems[18−20].

In this paper, robust state feedback synthesis problem for a class of uncertain polynomial systems is studied on the basis of SOS technique. Firstly, a nonlinear robust control approach via state feedback is presented. The approach strictly relies on SOS technique, because the settlement of uncertain terms depends on the extra independent variables, which are introduced to convert the robust control problems into SOS programming ones. Secondly, the approach of robust stabilization is extended to solve the robust optimization problems for the polynomial systems with performance objectives (guaranteed cost and H_∞ performance). Sufficient solvability conditions are given respectively for the nonlinear robust state feedback control problems with and without performance objectives. And all these solvability conditions are formulated as a constraint set of state dependent LMIs which can be solved by semidefinite programming relaxations based on SOS programming.

The main features of this paper are threefold. Firstly, by introducing SOS theory, an effective approach is provided to nonlinear robust synthesis problem for a class of polynomial systems with uncertainty and external disturbances. Secondly, the Lyapunov function matrix $P^{-1}(\boldsymbol{x})$ in the state dependent LMIs can be either polynomial or rational matrices, which potentially brings larger possibility to obtain the solutions to the problems than that in the existing work. Finally, the controllers in this paper are dependent on Lyapunov functions only, by which the solvability conditions can be easily extended to general polynomial Lyapunov functions, and the new solvability conditions can be derived by the computational method proposed in [15].

The remainder of this paper is organized as follows. In Section 1, some background materials related to SOS polynomial and the nonlinear robust control problem are provided. In Section 2, an SOS based approach is presented to solve the robust stabilization problem via state feedback. The guaranteed cost control as well as the H_∞ control is addressed in Section 3. In Section 4, a numerical example is given to illustrate the effectiveness of the approach. Section 5 concludes the paper.

1 Preliminaries

1.1 Sum of squares (SOS)

Definition 1 (SOS)[21]**.** A multivariate polynomial $p(x_1,\cdots,x_n) := p(\boldsymbol{x})$ is a sum of squares, if there exist polynomials $f_1(\boldsymbol{x}),\cdots,f_m(\boldsymbol{x})$ such that

$$p(\boldsymbol{x}) = \sum_{i=1}^{m} f_i^2(\boldsymbol{x})$$

Obviously, $p(\boldsymbol{x})$ being an SOS naturally implies $p(\boldsymbol{x}) \geq 0$ for all $\boldsymbol{x} \in \mathbf{R}^n$. In other words, the polynomial $p(\boldsymbol{x})$ is globally nonnegative if it can be formulated as a sum of squares, which provides a potentially effective criterion for analyzing the control problems of nonlinear systems. Even

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though the sum of squares condition is not necessary for nonnegativity, numerical experiments seem to indicate that the gap between them is not significant[22], for example, the nonnegativity and SOS are equivalent for the quadratic polynomials[23].

1.2 System description and problem statement

Consider the following uncertain nonlinear system in a polynomial form

$$\begin{aligned}\dot{\boldsymbol{x}} &= f(\boldsymbol{x}) + \Delta f(\boldsymbol{x}) + g_1(\boldsymbol{x})\boldsymbol{w} + (g_2(\boldsymbol{x}) + \Delta g_2(\boldsymbol{x}))\boldsymbol{u}\\ \boldsymbol{z} &= h(\boldsymbol{x}) + k(\boldsymbol{x})\boldsymbol{u}\end{aligned} \tag{1}$$

where $\boldsymbol{x} \in \mathbf{R}^n$ is the state, $\boldsymbol{u} \in \mathbf{R}^m$ is the control input, $\boldsymbol{w}(t) \in L_2[0,\infty)$ is an exogenous disturbance, $\boldsymbol{z} \in \mathbf{R}^r$ is the regulated output. Like the treatment in [14], system (1) can be rewritten in the following linear-like form

$$\begin{aligned}\dot{\boldsymbol{x}} &= (A(\boldsymbol{x}) + \Delta A(\boldsymbol{x}))\boldsymbol{Z}(\boldsymbol{x}) + B_1(\boldsymbol{x})\boldsymbol{w} + (B_2(\boldsymbol{x}) + \Delta B_2(\boldsymbol{x}))\boldsymbol{u}\\ \boldsymbol{z} &= C(\boldsymbol{x})\boldsymbol{Z}(\boldsymbol{x}) + D(\boldsymbol{x})\boldsymbol{u}\end{aligned} \tag{2}$$

where $\boldsymbol{Z}(\boldsymbol{x})$ is an $N \times 1$ vector of monomials in $\boldsymbol{x}$, $A(\boldsymbol{x})$, $B_1(\boldsymbol{x})$, $B_2(\boldsymbol{x})$, $C(\boldsymbol{x})$, $D(\boldsymbol{x})$ are known polynomial matrices with proper dimensions, $\Delta A(\boldsymbol{x})$ and $\Delta B_2(\boldsymbol{x})$ are uncertain matrices.

For further use, the following assumptions are required.

Assumption 1[14]**.** $\boldsymbol{Z}(\boldsymbol{x}) = \boldsymbol{0}$ if and only if $\boldsymbol{x} = \boldsymbol{0}$.

Assumption 2[2]**.** $D^{\mathrm{T}}(\boldsymbol{x})[C(\boldsymbol{x})\ D(\boldsymbol{x})] = [0\ \ I]$.

Assumption 3[24]**.** $[\Delta A(\boldsymbol{x})\ \Delta B_2(\boldsymbol{x})] = E(\boldsymbol{x})\Delta(\boldsymbol{x},t)\times [G_1(\boldsymbol{x})\ G_2(\boldsymbol{x})]$, and the $\Delta(\boldsymbol{x},t)$ satisfies

$$\int_0^\infty (||\Phi(\boldsymbol{x})||^2 - ||\Delta(\boldsymbol{x},t)\Phi(\boldsymbol{x})||^2)\mathrm{d}t \ge 0 \tag{3}$$

for any known matrix function $\Phi(\boldsymbol{x})$, where $E(\boldsymbol{x})$, $G_1(\boldsymbol{x})$, $G_2(\boldsymbol{x})$ are known polynomial matrices.

Assumption 4. $G_2^{\mathrm{T}}(\boldsymbol{x})G_2(\boldsymbol{x})$ is nonsingular.

For future deduction, define $M(\boldsymbol{x})$ to be an $N \times n$ polynomial matrix whose (i,j)-th entry is given by

$$M_{ij}(\boldsymbol{x}) = \frac{\partial Z_i}{\partial x_j}(\boldsymbol{x})$$

for $i = 1,\cdots,N$, $j = 1,\cdots,n$. Let $A_j(\boldsymbol{x})$, $\Delta A_j(\boldsymbol{x})$, $B_{2j}(\boldsymbol{x})$, $\Delta B_{2j}(\boldsymbol{x})$, $E_j(\boldsymbol{x})$ denote the j-th row of $A(\boldsymbol{x})$, $\Delta A(\boldsymbol{x})$, $B_2(\boldsymbol{x})$, $\Delta B_2(\boldsymbol{x})$, $E(\boldsymbol{x})$, respectively. $J = \{j_1,\cdots,j_m\}$ denote the row indices of $[B_1(\boldsymbol{x})\quad B_2(\boldsymbol{x})]$ whose corresponding row is equal to zero, and define $E_J(\boldsymbol{x}) = [E_{j_1}^{\mathrm{T}}(\boldsymbol{x}),\cdots,E_{j_m}^{\mathrm{T}}(\boldsymbol{x})]^{\mathrm{T}}$, $\tilde{\boldsymbol{x}} = [x_{j_1},\cdots,x_{j_m}]^{\mathrm{T}}$.

Assumption 5. $\Delta B_{2j_i}(\boldsymbol{x})$ is zero, $j_i \in J$.

Remark 1. Note that Assumption 2 is a standard H_∞ assumption[2] and the constraint given by (3) means that the uncertainty satisfies $||\Delta(\boldsymbol{x},t)||^2 \le 1$.

Definition 2[5]**.** The system $P: \begin{cases}\dot{\boldsymbol{x}} = f(\boldsymbol{x}) + g(\boldsymbol{x})\boldsymbol{w}\\ \boldsymbol{z} = h(\boldsymbol{x}) + k(\boldsymbol{x})\boldsymbol{w}\end{cases}$ with initial condition $\boldsymbol{x}(0) = \boldsymbol{0}$ is said to have L_2-gain less than or equal to γ for $\gamma > 0$ if

$$\int_0^T ||\boldsymbol{z}(t)||^2\mathrm{d}t \le \gamma^2\int_0^T ||\boldsymbol{w}(t)||^2\mathrm{d}t$$

for all $T \ge 0$ and $\boldsymbol{w}(t) \in L_2[0,T]$.

The nonlinear state feedback H_∞ control problem is concerned with finding a state feedback controller such that the closed loop system is internally stable, i.e. asymptotically stable when $\boldsymbol{w}(t) \equiv \boldsymbol{0}$ and L_2-gain$\le \gamma$[1]. The latter means that the L_2-gain of the mapping from the exogenous input disturbance $\boldsymbol{w}(t) \in L_2[0,\infty)$ to the regulated output $\boldsymbol{z}$ is guaranteed to be less than or equal to γ. Then, the corresponding robust H_∞ control problem can be stated by finding a state feedback controller under which the closed loop system is internally stable and has L_2-gain$\le \gamma$ subject to any admissible uncertainty.

Remark 2. L_2-gain$\le\gamma$ is a necessary condition for H_∞ control problem. This gain restriction plus the so-called zero-state detectable condition can guarantee the internal stability of a system, thus the H_∞ control problem of the system is solvable[1]. As is well known, the zero-state detectable is usually uneasy to certify for a nonlinear system, whereas the internal stability can be satisfied with the aid of SOS constraints. In this paper, the H_∞ control problem is studied without the assumption of the zero-state detectability for polynomial systems because the internal stability is ensured by SOS based conditions.

For the system described in (2), the nonlinear robust control synthesis problem in this paper mainly involves two aspects, i.e., 1) to propose an approach to design a state feedback controller $\boldsymbol{u}(\boldsymbol{x})$ under which the zero equilibrium of the closed loop system of (2) is asymptotically stable; 2) to extend the approach to the analysis of nonlinear robust optimization problems with (optimal) guaranteed cost and H_∞ performance objectives respectively.

2 State feedback stabilization

In this section, state feedback stabilization against any admissible uncertainty will be considered. The objective is designing a robust controller to stabilize system (2) under the case that the disturbance vanishes.

Before discussing the state feedback synthesis problem, the following proposition and lemmas are presented.

Proposition 1. Let $L(\boldsymbol{x}) \in \mathbf{R}^{n\times n}$ be a nonsingular polynomial matrix for all $\boldsymbol{x}$. There exist a polynomial matrix $\Theta(\boldsymbol{x}) \in \mathbf{R}^{n\times n}$ and a polynomial $\tau(\boldsymbol{x})$ satisfying $\Theta(\boldsymbol{x}) = c(\boldsymbol{x})L^*(\boldsymbol{x})$ and $\tau(\boldsymbol{x}) = c(\boldsymbol{x})\det(L(\boldsymbol{x}))$, then $L^{-1}(\boldsymbol{x})$ can be described as $L^{-1}(\boldsymbol{x}) = \tau^{-1}(\boldsymbol{x})\Theta(\boldsymbol{x})$, where $L^*(\boldsymbol{x})$ is the adjoint matrix of $L(\boldsymbol{x})$, which is also a polynomial matrix, $c(\boldsymbol{x})$ is a nonzero polynomial.

Proof. Since $L(\boldsymbol{x})$ is nonsingular, one has $\det(L(\boldsymbol{x})) = d(\boldsymbol{x}) \ne 0$ and $L(\boldsymbol{x})L^{-1}(\boldsymbol{x}) = I$. By the equality

$$L(\boldsymbol{x})L^*(\boldsymbol{x}) = L^*(\boldsymbol{x})L(\boldsymbol{x}) = d(\boldsymbol{x})I$$

which means $L(\boldsymbol{x})(d^{-1}(\boldsymbol{x})L^*(\boldsymbol{x})) = I$, $L^{-1}(\boldsymbol{x})$ can be expressed as $L^{-1}(\boldsymbol{x}) = d^{-1}(\boldsymbol{x})L^*(\boldsymbol{x})$. Therefore,

$$L^{-1}(\boldsymbol{x}) = \frac{c(\boldsymbol{x})L^*(\boldsymbol{x})}{d(\boldsymbol{x})c(\boldsymbol{x})} = \frac{\Theta(\boldsymbol{x})}{\tau(\boldsymbol{x})}$$

This immediately gives the result of Proposition 1. □

Remark 3. If $d(\boldsymbol{x})$ is a constant, $\tau(\boldsymbol{x})$ and $\Theta(\boldsymbol{x})$ can be chosen as $\tau(\boldsymbol{x}) = 1$ and $\Theta(\boldsymbol{x}) = d^{-1}(\boldsymbol{x})L^*(\boldsymbol{x})$. Without loss of generality, $\tau(\boldsymbol{x}) > 0$ is always supposed. For example, in Section 2, let $P^{-1}(\tilde{\boldsymbol{x}}) = q^{-1}(\tilde{\boldsymbol{x}})Q(\tilde{\boldsymbol{x}})$, $q^{-1}(\tilde{\boldsymbol{x}}) > 0$. And $\tau(\boldsymbol{x}) > 0$ can be guaranteed by introducing the polynomial $c(\boldsymbol{x})$ (i.e., let $c(\boldsymbol{x}) = d(\boldsymbol{x})$).

Lemma 1[14]**.** For a symmetric polynomial matrix $P(\boldsymbol{x})$ which is nonsingular for all $\boldsymbol{x}$, then

$$\frac{\partial P}{\partial x_i}(\boldsymbol{x}) = -P(\boldsymbol{x})\frac{\partial P^{-1}}{\partial x_i}(\boldsymbol{x})P(\boldsymbol{x})$$

Lemma 2[25]**.** Given arbitrary vectors $\boldsymbol{x} \in \mathbf{R}^p$, $\boldsymbol{y} \in \mathbf{R}^q$, one has

$$\max\{(\boldsymbol{x}^{\mathrm{T}}F\boldsymbol{y})^2 : F \in \mathbf{R}^{p\times q}, F^{\mathrm{T}}F \le I\} = (\boldsymbol{x}^{\mathrm{T}}\boldsymbol{x})(\boldsymbol{y}^{\mathrm{T}}\boldsymbol{y})$$

Suppose $B_1(\boldsymbol{x}) = 0$, disregarding $\boldsymbol{z}$ and $\boldsymbol{w}$, then system (2) can be rewritten as

$$\dot{\boldsymbol{x}} = (A(\boldsymbol{x}) + \Delta A(\boldsymbol{x}))\boldsymbol{Z}(\boldsymbol{x}) + (B_2(\boldsymbol{x}) + \Delta B_2(\boldsymbol{x}))\boldsymbol{u} \tag{4}$$

Theorem 1. Suppose system (4) satisfies Assumptions 1, 3, 4 and 5. There exist an $N\times N$ matrix $Q(\tilde{\boldsymbol{x}})$, a constant $\varepsilon > 0$, and some SOS polynomials $\varepsilon_1(\boldsymbol{x}) > 0$, $\varepsilon_2(\boldsymbol{x}) > 0$, $q(\tilde{\boldsymbol{x}}) > 0$, $s(\boldsymbol{x}) \geq 0$, $s_1(\boldsymbol{x}) > 0$ such that

$$\boldsymbol{v}_1{}^{\mathrm{T}}(Q(\tilde{\boldsymbol{x}}) - \varepsilon I)\boldsymbol{v}_1 \quad \text{is an SOS} \tag{5}$$

$$-\boldsymbol{v}_2^{\mathrm{T}}\Xi_Q(\boldsymbol{x})\boldsymbol{v}_2 - \boldsymbol{v}_a^{\mathrm{T}}s(\boldsymbol{x})\Pi(\boldsymbol{x})\boldsymbol{v}_a \quad \text{is an SOS} \tag{6}$$

$$\boldsymbol{v}_3^{\mathrm{T}}\begin{bmatrix} s^2(\boldsymbol{x})(\boldsymbol{v}_a^{\mathrm{T}}\boldsymbol{v}_a)^2 & \boldsymbol{\varphi}(\tilde{\boldsymbol{x}})E_J(\boldsymbol{x}) \\ * & I\end{bmatrix}\boldsymbol{v}_3 \quad \text{is an SOS} \tag{7}$$

then the state feedback stabilization problem is solvable, and the controller is given by

$$\boldsymbol{u}(\boldsymbol{x}) = -\tau_1^{-1}(\boldsymbol{x})\Theta_1(\boldsymbol{x})q(\tilde{\boldsymbol{x}})B_2^{\mathrm{T}}(\boldsymbol{x})M^{\mathrm{T}}(\boldsymbol{x})Q^{-1}(\tilde{\boldsymbol{x}})\boldsymbol{Z}(\boldsymbol{x})$$

where $\boldsymbol{v}_1$, $\boldsymbol{v}_2 = [\boldsymbol{v}_a^{\mathrm{T}}, \boldsymbol{v}_b^{\mathrm{T}}]^{\mathrm{T}}$, $\boldsymbol{v}_3$ are column vectors with proper dimensions, $\boldsymbol{v}_a \in \mathbf{R}^N$, $\boldsymbol{v}_b \in \mathbf{R}^{\dim(\boldsymbol{v}_2)-N}$, $\Pi(\boldsymbol{x}) = \boldsymbol{Z}^{\mathrm{T}}(\boldsymbol{x}) \times G^{\mathrm{T}}(\boldsymbol{x})G(\boldsymbol{x})\boldsymbol{Z}(\boldsymbol{x}) + 1$, $\Xi_Q(\boldsymbol{x}) = \begin{bmatrix}\Psi_q(\tilde{\boldsymbol{x}},\boldsymbol{x}) & * \\ \tau_1(\boldsymbol{x})G_1(\boldsymbol{x})Q(\tilde{\boldsymbol{x}}) & -\varepsilon_1(\boldsymbol{x})I\end{bmatrix}$, $\Psi_q(\tilde{\boldsymbol{x}},\boldsymbol{x}) = \tau_1^2(\boldsymbol{x})[q(\tilde{\boldsymbol{x}})Q(\tilde{\boldsymbol{x}})A^{\mathrm{T}}(\boldsymbol{x})M^{\mathrm{T}}(\boldsymbol{x}) + q(\tilde{\boldsymbol{x}})M(\boldsymbol{x}) \times A(\boldsymbol{x})Q(\tilde{\boldsymbol{x}}) + q^2(\tilde{\boldsymbol{x}})(\varepsilon_1(\boldsymbol{x}) + \varepsilon_2(\boldsymbol{x}))M(\boldsymbol{x})E(\boldsymbol{x})E^{\mathrm{T}}(\boldsymbol{x})M^{\mathrm{T}}(\boldsymbol{x})] - \varepsilon_2(\boldsymbol{x})q^2(\tilde{\boldsymbol{x}})\tau_1(\boldsymbol{x})M(\boldsymbol{x})B_2(\boldsymbol{x})\Theta_1(\boldsymbol{x})B_2^{\mathrm{T}}(\boldsymbol{x})M^{\mathrm{T}}(\boldsymbol{x}) - \tau_1^2(\boldsymbol{x}) \times \sum_{j\in J}(q(\tilde{\boldsymbol{x}})\frac{\partial Q(\tilde{\boldsymbol{x}})}{\partial x_j} - Q(\tilde{\boldsymbol{x}})\frac{\partial q(\tilde{\boldsymbol{x}})}{\partial x_j})(A_j(\boldsymbol{x})\boldsymbol{Z}(\boldsymbol{x})) + s_1(\boldsymbol{x})I$, $\boldsymbol{\varphi}(\tilde{\boldsymbol{x}}) = \tau_1^2(\boldsymbol{x})\left[\boldsymbol{v}_a^{\mathrm{T}}(q(\tilde{\boldsymbol{x}})\frac{\partial Q(\tilde{\boldsymbol{x}})}{\partial x_{j_1}} - Q(\tilde{\boldsymbol{x}})\frac{\partial q(\tilde{\boldsymbol{x}})}{\partial x_{j_1}})\boldsymbol{v}_a, \cdots, \boldsymbol{v}_a^{\mathrm{T}}(q(\tilde{\boldsymbol{x}})\frac{\partial Q(\tilde{\boldsymbol{x}})}{\partial x_{j_m}} - Q(\tilde{\boldsymbol{x}})\frac{\partial q(\tilde{\boldsymbol{x}})}{\partial x_{j_m}})\boldsymbol{v}_a\right]$. $\tau_1(\boldsymbol{x})$ and $\Theta_1(\boldsymbol{x})$ can be obtained from Proposition 1 by $L(\boldsymbol{x}) = G_2^{\mathrm{T}}(\boldsymbol{x})G_2(\boldsymbol{x})$.

Proof. Motivated by [14], define the Lyapunov function candidate as follows

$$V(\boldsymbol{x}) = \boldsymbol{Z}^{\mathrm{T}}(\boldsymbol{x})P(\tilde{\boldsymbol{x}})\boldsymbol{Z}(\boldsymbol{x})$$

where $P(\tilde{\boldsymbol{x}})$ is a positive definite matrix.

Obviously, Assumption 1 and $P(\tilde{\boldsymbol{x}}) > 0$ imply that $V(\boldsymbol{0}) = 0$ and $V(\boldsymbol{x}) > 0$ for $\boldsymbol{x} \neq \boldsymbol{0}$.

Note that system (4) satisfies Assumption 3. Referring to linear system robust control theory, the following conclusion can be obtained

$$\dot{V}(\boldsymbol{x}) \leq \boldsymbol{Z}^{\mathrm{T}}(\boldsymbol{x})\mathscr{F}_P(P,\boldsymbol{x})\boldsymbol{Z}(\boldsymbol{x}) + \mathscr{F}_{PU}(P,\boldsymbol{x},\boldsymbol{u}) \tag{8}$$

where $\varepsilon_1(\boldsymbol{x})$, $\varepsilon_2(\boldsymbol{x})$ are positive polynomials, $\mathscr{F}_P(P,\boldsymbol{x}) = A^{\mathrm{T}}(\boldsymbol{x})M^{\mathrm{T}}(\boldsymbol{x})P(\tilde{\boldsymbol{x}}) + P(\tilde{\boldsymbol{x}})M(\boldsymbol{x})A(\boldsymbol{x}) + (\varepsilon_1(\boldsymbol{x}) + \varepsilon_2(\boldsymbol{x})) \times P(\tilde{\boldsymbol{x}})M(\boldsymbol{x})E(\boldsymbol{x})E^{\mathrm{T}}(\boldsymbol{x})M^{\mathrm{T}}(\boldsymbol{x})P(\tilde{\boldsymbol{x}}) + \varepsilon_1^{-1}(\boldsymbol{x})G_1^{\mathrm{T}}(\boldsymbol{x})G_1(\boldsymbol{x}) + \sum_{j\in J}\frac{\partial P(\tilde{\boldsymbol{x}})}{\partial x_j}(A_j(\boldsymbol{x})\boldsymbol{Z}(\boldsymbol{x})) + \sum_{j\in J}\frac{\partial P(\tilde{\boldsymbol{x}})}{\partial x_j}(\Delta A_j(\boldsymbol{x})\boldsymbol{Z}(\boldsymbol{x}))$, $\mathscr{F}_{PU}(P,\boldsymbol{x},\boldsymbol{u}) = 2\boldsymbol{Z}^{\mathrm{T}}(\boldsymbol{x})P(\tilde{\boldsymbol{x}})M(\boldsymbol{x})B_2(\boldsymbol{x})\boldsymbol{u} + \boldsymbol{u}^{\mathrm{T}}\Gamma_1(\boldsymbol{x})\boldsymbol{u}$, $\Gamma_1(\boldsymbol{x}) = \varepsilon_2^{-1}(\boldsymbol{x})G_2^{\mathrm{T}}(\boldsymbol{x})G_2(\boldsymbol{x})$.

In order to facilitate controller design, $\mathscr{F}_{PU}(P,\boldsymbol{x},\boldsymbol{u})$ can be rewritten as

$$\begin{aligned}\mathscr{F}_{PU}(P,\boldsymbol{x},\boldsymbol{u}) =& (\boldsymbol{u} - k(\boldsymbol{x}))^{\mathrm{T}}\Gamma_1(\boldsymbol{x})(\boldsymbol{u} - k(\boldsymbol{x})) - \\ & k^{\mathrm{T}}(\boldsymbol{x})\Gamma_1(\boldsymbol{x})k(\boldsymbol{x})\end{aligned} \tag{9}$$

where $k(\boldsymbol{x}) = -\Gamma_1^{-1}(\boldsymbol{x})B_2^{\mathrm{T}}(\boldsymbol{x})M^{\mathrm{T}}(\boldsymbol{x})P(\tilde{\boldsymbol{x}})\boldsymbol{Z}(\boldsymbol{x})$.

Let $\boldsymbol{u} = k(\boldsymbol{x})$, (9) becomes

$$\begin{aligned}&\mathscr{F}_{PU}(P,\boldsymbol{x},\boldsymbol{u}) = \\ &-\boldsymbol{Z}^{\mathrm{T}}(\boldsymbol{x})P(\tilde{\boldsymbol{x}})M(\boldsymbol{x})B_2(\boldsymbol{x})\Gamma_1^{-1}(\boldsymbol{x})B_2^{\mathrm{T}}(\boldsymbol{x})M^{\mathrm{T}}(\boldsymbol{x})P(\tilde{\boldsymbol{x}})\boldsymbol{Z}(\boldsymbol{x})\end{aligned}$$

Therefore, (8) is equivalent to

$$\dot{V}(\boldsymbol{x}) \leq \boldsymbol{Z}^{\mathrm{T}}(\boldsymbol{x})\mathscr{F}_{P\Gamma}(P,\boldsymbol{x},\Gamma_1)\boldsymbol{Z}(\boldsymbol{x}) \tag{10}$$

where $\mathscr{F}_{P\Gamma}(P,\boldsymbol{x},\Gamma_1) = \mathscr{F}_P(P,\boldsymbol{x}) - P(\tilde{\boldsymbol{x}})M(\boldsymbol{x})B_2(\boldsymbol{x})\Gamma_1^{-1}(\boldsymbol{x})B_2^{\mathrm{T}}(\boldsymbol{x})M^{\mathrm{T}}(\boldsymbol{x})P(\tilde{\boldsymbol{x}})$.

By Proposition 1, (10) can be represented as

$$\dot{V}(\boldsymbol{x}) \leq \boldsymbol{Z}^{\mathrm{T}}(\boldsymbol{x})\mathscr{F}_{P\Theta}(P,\boldsymbol{x},\Theta_1)\boldsymbol{Z}(\boldsymbol{x}) \tag{11}$$

where $\mathscr{F}_{P\Theta}(P,\boldsymbol{x},\Theta_1) = \mathscr{F}_P(P,\boldsymbol{x}) - \varepsilon_2(\boldsymbol{x})\tau_1^{-1}(\boldsymbol{x})P(\tilde{\boldsymbol{x}})M(\boldsymbol{x})B_2(\boldsymbol{x})\Theta_1(\boldsymbol{x})B_2^{\mathrm{T}}(\boldsymbol{x})M^{\mathrm{T}}(\boldsymbol{x})P(\tilde{\boldsymbol{x}})$, $\tau_1(\boldsymbol{x}) = \det(\varepsilon_2(\boldsymbol{x})\Gamma_1(\boldsymbol{x}))$, $\Theta_1(\boldsymbol{x}) = (\varepsilon_2(\boldsymbol{x})\Gamma_1(\boldsymbol{x}))^*$.

Thus, (11) implies that

$$\dot{V}(\boldsymbol{x}) < 0 \text{ if } \mathscr{F}_{P\Theta}(P,\boldsymbol{x},\Theta_1) < 0$$

Multiplying the $\mathscr{F}_{P\Theta}(P,\boldsymbol{x},\Theta_1)$ from the left and right by $P^{-1}(\tilde{\boldsymbol{x}})$, and using the result in Lemma 1, a new matrix $\mathscr{F}_{P^{-1}\Theta}(P^{-1},\boldsymbol{x},\Theta_1)$ can be obtained. By Proposition 1, let $P^{-1}(\tilde{\boldsymbol{x}}) = q^{-1}(\tilde{\boldsymbol{x}})Q(\tilde{\boldsymbol{x}})$, where $q(\tilde{\boldsymbol{x}}) > 0$ is a polynomial, and $Q(\tilde{\boldsymbol{x}})$ is a polynomial matrix. It is easy to conclude that $\dot{V}(\boldsymbol{x}) < 0$ holds, if

$$\mathscr{F}_Q(Q,\boldsymbol{x},\Theta_1) := q^2(\tilde{\boldsymbol{x}})\mathscr{F}_{P^{-1}\Theta}(q^{-1}Q,\boldsymbol{x},\Theta_1) < 0 \tag{12}$$

where

$$\begin{aligned}\mathscr{F}_Q(Q,\boldsymbol{x},\Theta_1) =& q(\tilde{\boldsymbol{x}})[Q(\tilde{\boldsymbol{x}})A^{\mathrm{T}}(\boldsymbol{x})M^{\mathrm{T}}(\boldsymbol{x}) + M(\boldsymbol{x})A(\boldsymbol{x})Q(\tilde{\boldsymbol{x}})] + \\ & q^2(\tilde{\boldsymbol{x}})[(\varepsilon_1(\boldsymbol{x}) + \varepsilon_2(\boldsymbol{x}))M(\boldsymbol{x})E(\boldsymbol{x})E^{\mathrm{T}}(\boldsymbol{x})M^{\mathrm{T}}(\boldsymbol{x}) - \\ & \varepsilon_2(\boldsymbol{x})\tau_1^{-1}(\boldsymbol{x})M(\boldsymbol{x})B_2(\boldsymbol{x})\Theta_1(\boldsymbol{x})B_2^{\mathrm{T}}(\boldsymbol{x})M^{\mathrm{T}}(\boldsymbol{x})] + \\ & \varepsilon_1^{-1}(\boldsymbol{x})Q(\tilde{\boldsymbol{x}})G_1^{\mathrm{T}}(\boldsymbol{x})G_1(\boldsymbol{x})Q(\tilde{\boldsymbol{x}}) - \sum_{j\in J}[(q(\tilde{\boldsymbol{x}})\frac{\partial Q(\tilde{\boldsymbol{x}})}{\partial x_j} - \\ & Q(\tilde{\boldsymbol{x}})\frac{\partial q(\tilde{\boldsymbol{x}})}{\partial x_j})((A_j(\boldsymbol{x}) + \Delta A_j(\boldsymbol{x}))\boldsymbol{Z}(\boldsymbol{x}))]\end{aligned}$$

A matrix inequality can be obtained by using Schur complement to $\mathscr{F}_Q(Q,\boldsymbol{x},\Theta_1) < 0$ and multiplying the matrix inequality from left and right by $\mathrm{diag}\{\tau_1\boldsymbol{x})I, I\}$, it is easy to obtain that $\mathscr{F}_Q(Q,\boldsymbol{x},\Theta_1) < 0$, if

$$-\boldsymbol{v}_2^{\mathrm{T}}\Xi_Q(\boldsymbol{x})\boldsymbol{v}_2 + \boldsymbol{v}_2^{\mathrm{T}}\begin{bmatrix}\mathscr{F}_{\partial Q}(Q,\boldsymbol{x}) & \bar{0}_1 \\ \bar{0}_1^{\mathrm{T}} & \bar{0}_2\end{bmatrix}\boldsymbol{v}_2 \geq 0 \tag{13}$$

where $\bar{0}_1$, $\bar{0}_2$ are zero matrices with proper dimensions, $\mathscr{F}_{\partial Q}(Q,\boldsymbol{x}) = \tau_1^2(\boldsymbol{x})\sum_{j\in J}(q(\tilde{\boldsymbol{x}})\frac{\partial Q(\tilde{\boldsymbol{x}})}{\partial x_j} - Q(\tilde{\boldsymbol{x}})\frac{\partial q(\tilde{\boldsymbol{x}})}{\partial x_j})(\Delta A_j(\boldsymbol{x})\times \boldsymbol{Z}(\boldsymbol{x}))$ then (13) can be rewritten as

$$-\boldsymbol{v}_2^{\mathrm{T}}\Xi_Q(\boldsymbol{x})\boldsymbol{v}_2 + \boldsymbol{\varphi}(\tilde{\boldsymbol{x}})E_J(\boldsymbol{x})\Delta(\boldsymbol{x},t)G_1(\boldsymbol{x})\boldsymbol{Z}(\boldsymbol{x}) \geq 0 \tag{14}$$

Note that (7) implies

$$\boldsymbol{\varphi}(\tilde{\boldsymbol{x}})E_J(\boldsymbol{x})E_J^{\mathrm{T}}(\boldsymbol{x})\boldsymbol{\varphi}^{\mathrm{T}}(\tilde{\boldsymbol{x}}) \leq (\boldsymbol{v}_a^{\mathrm{T}}s(\boldsymbol{x})\boldsymbol{v}_a)^2 \tag{15}$$

Combining (14), (15) with Lemma 2,

$$\begin{aligned}&-\boldsymbol{v}_2^{\mathrm{T}}\Xi_Q(\boldsymbol{x})\boldsymbol{v}_2 + \boldsymbol{\varphi}(\tilde{\boldsymbol{x}})E_J(\boldsymbol{x})\Delta(\boldsymbol{x},t)G_1(\boldsymbol{x})\boldsymbol{Z}(\boldsymbol{x}) \geq \\ &-\boldsymbol{v}_2^{\mathrm{T}}\Xi_Q(\boldsymbol{x})\boldsymbol{v}_2 - |\boldsymbol{\varphi}(\tilde{\boldsymbol{x}})E_J(\boldsymbol{x})||G_1(\boldsymbol{x})\boldsymbol{Z}(\boldsymbol{x})| \geq \\ &-\boldsymbol{v}_2^{\mathrm{T}}\Xi_Q(\boldsymbol{x})\boldsymbol{v}_2 - \boldsymbol{v}_a^{\mathrm{T}}s(\boldsymbol{x})|G_1(\boldsymbol{x})\boldsymbol{Z}(\boldsymbol{x})|\boldsymbol{v}_a \geq \\ &-\boldsymbol{v}_2^{\mathrm{T}}\Xi_Q(\boldsymbol{x})\boldsymbol{v}_2 - \boldsymbol{v}_a^{\mathrm{T}}s(\boldsymbol{x})\Pi(\boldsymbol{x})\boldsymbol{v}_a\end{aligned} \tag{16}$$

where $\Pi(\boldsymbol{x}) = \boldsymbol{Z}^{\mathrm{T}}(\boldsymbol{x})G_1^{\mathrm{T}}(\boldsymbol{x})G_1(\boldsymbol{x})\boldsymbol{Z}(\boldsymbol{x}) + 1$.

By (16), it is clear that (6) and (7) imply $\dot{V}(\boldsymbol{x}) < 0$. Thus, if (5) ~ (7) hold, the zero equilibrium of the closed loop system is asymptotically stable. □

Remark 4. If $P(\tilde{\boldsymbol{x}})$ in Theorem 1 is a constant matrix, the stability of the closed loop system holds globally[14].

Remark 5. Furthermore, if the constraint (7) holds with $s(\boldsymbol{x}) = 0$, from (15), $\boldsymbol{\varphi}(\tilde{\boldsymbol{x}})E_J(\boldsymbol{x}) = 0$, which means that there exists a Lyapunov function matrix $P(\tilde{\boldsymbol{x}})$ such that $\boldsymbol{\varphi}(\tilde{\boldsymbol{x}})E_J(\boldsymbol{x})\Delta(\boldsymbol{x},t)G_1(\boldsymbol{x})\boldsymbol{Z}(\boldsymbol{x}) = 0$. In this case, (16) is unnecessary, and the second term in (6) is zero, which shows that the treatment of $\boldsymbol{\varphi}(\tilde{\boldsymbol{x}})E_J(\boldsymbol{x})\Delta(\boldsymbol{x},t)G_1(\boldsymbol{x})\boldsymbol{Z}(\boldsymbol{x})$ do not have any conservatism.

Remark 6. Using SOS technique to deal with the control problems with Lyapunov function of the form $V(\boldsymbol{x}) = \boldsymbol{Z}^{\mathrm{T}}(\boldsymbol{x})P(\tilde{\boldsymbol{x}})\boldsymbol{Z}(\boldsymbol{x})$ (or $P(\boldsymbol{x})$) in the existing approaches, a limitation is that $P^{-1}(\tilde{\boldsymbol{x}})$ must be a polynomial matrix[14,16] however, in this paper, $P^{-1}(\tilde{\boldsymbol{x}})$ can also be a rational matrix, which relies on $q(\tilde{\boldsymbol{x}})$. If $q(\tilde{\boldsymbol{x}}) = 1$, the same form of the Lyapunov function as in [14, 16] can be obtained, which means that the Lyapunov function constructed by the method proposed in the paper includes the function constructed in [14, 16]. Thus, the analysis conditions in this paper are potentially more flexible.

From the proof of Theorem 1, it is easy to observe that $s(\boldsymbol{x})$ plays an important role in the settlement of the uncertain term $\sum_{j\in J}\frac{\partial Q(\tilde{\boldsymbol{x}})}{\partial x_j}(\Delta A_j(\boldsymbol{x})\boldsymbol{Z}(\boldsymbol{x}))$. Therefore, the optimal $s(\boldsymbol{x})$ will reduce the conservatism of the condition. The following corollary is given to optimize $s(\boldsymbol{x})$.

Corollary 1. Suppose system (4) satisfies Assumptions 1, 3, 4, and 5. There exist an $N\times N$ matrix $Q(\tilde{\boldsymbol{x}})$, a constant $\varepsilon > 0$, and some SOS polynomials $\varepsilon_1(\boldsymbol{x}) > 0$, $\varepsilon_2(\boldsymbol{x}) > 0$, $q(\tilde{\boldsymbol{x}}) > 0$, $s(\boldsymbol{x}) \geq 0$, $s_1(\boldsymbol{x}) > 0$ such that the following SOS optimization problem

min ρ

s.t.

$$\boldsymbol{v}_1^{\mathrm{T}}(Q(\tilde{\boldsymbol{x}}) - \varepsilon I)\boldsymbol{v}_1 \quad \text{is an SOS} \tag{17}$$

$$-\boldsymbol{v}_2^{\mathrm{T}}\Xi_Q(\boldsymbol{x})\boldsymbol{v}_2 - \boldsymbol{v}_a^{\mathrm{T}}\rho s(\boldsymbol{x})\Pi(\boldsymbol{x})\boldsymbol{v}_a \quad \text{is an SOS} \tag{18}$$

$$\boldsymbol{v}_3^{\mathrm{T}}\begin{bmatrix}\rho s^2(\boldsymbol{x})(\boldsymbol{v}_a^{\mathrm{T}}\boldsymbol{v}_a)^2 & \boldsymbol{\varphi}(\tilde{\boldsymbol{x}})E_J(\boldsymbol{x})\\ * & \rho I\end{bmatrix}\boldsymbol{v}_3 \quad \text{is an SOS} \tag{19}$$

is solvable. Then for the controller,

$$\boldsymbol{u}(\boldsymbol{x}) = -\tau_1^{-1}(\boldsymbol{x})\Theta_1(\boldsymbol{x})q(\tilde{\boldsymbol{x}})B_2^{\mathrm{T}}(\boldsymbol{x})M^{\mathrm{T}}(\boldsymbol{x})Q^{-1}(\tilde{\boldsymbol{x}})\boldsymbol{Z}(\boldsymbol{x})$$

the zero equilibrium of the closed loop system is asymptotically stable, where all parameters are the same as in Theorem 1.

Proof. Referring to the proof of Theorem 1, suppose

$$\boldsymbol{\varphi}(\tilde{\boldsymbol{x}})E_J(\boldsymbol{x})E_J^{\mathrm{T}}(\boldsymbol{x})\boldsymbol{\varphi}^{\mathrm{T}}(\tilde{\boldsymbol{x}}) \leq (\boldsymbol{v}_a^{\mathrm{T}}\rho s(\boldsymbol{x})\boldsymbol{v}_a)^2 \tag{20}$$

A matrix inequality can be obtained by using Schur complement to (20), and multiplying the matrix inequality from left and right by $\mathrm{diag}\{\rho^{-1/2}I, \rho^{1/2}I\}$, then (20) is equivalent to

$$\begin{bmatrix}\rho s^2(\boldsymbol{x})(\boldsymbol{v}_a^{\mathrm{T}}\boldsymbol{v}_a)^2 & \boldsymbol{\varphi}(\tilde{\boldsymbol{x}})E_J(\boldsymbol{x})\\ * & \rho I\end{bmatrix} \geq 0 \tag{21}$$

The rest of the proof can be obtained by following the proof of Theorem 1. □

Remark 7. If the optimum of ρ is zero, which means $\boldsymbol{\varphi}(\tilde{\boldsymbol{x}})E_J(\boldsymbol{x}) = 0$, the same conclusion as Remark 5 can be obtained.

3 State feedback synthesis with performance objectives

In this section, the robust control approach to uncertain nonlinear systems proposed in Theorem 1 is used to analyze the nonlinear optimization problems with (optimal) guaranteed cost and H_∞ performance objectives.

3.1 Nonlinear optimal control

The optimal control problem of system (2) being considered is the extension of the linear quadratic regulator (LQR) problem to the nonlinear systems[26]. The main work of this subsection is to compute an upper bound c on the two-norm of the output signal (i.e., $||\boldsymbol{z}||_2^2 < c$), augmenting the minimization of c. The problem with such performance objective is also discussed in [10, 12, 14], but in [12], it is known as guaranteed cost control problem. To this end, define the output energy as

$$||\boldsymbol{z}||_2^2 = \lim_{T\to\infty}\int_0^T \boldsymbol{z}^{\mathrm{T}}\boldsymbol{z}\mathrm{d}t \tag{22}$$

For simplicity, suppose that system (2) without regard to disturbance $\boldsymbol{w}$ ($B_1(\boldsymbol{x}) = 0$) is of the form

$$\begin{aligned}&\dot{\boldsymbol{x}} = (A(\boldsymbol{x}) + \Delta A(\boldsymbol{x}))\boldsymbol{Z}(\boldsymbol{x}) + (B_2(\boldsymbol{x}) + \Delta B_2(\boldsymbol{x}))\boldsymbol{u}\\ &\boldsymbol{z} = C(\boldsymbol{x})\boldsymbol{Z}(\boldsymbol{x}) + D(\boldsymbol{x})\boldsymbol{u}\\ &\boldsymbol{x}(0) = \boldsymbol{x}_0\end{aligned} \tag{23}$$

Theorem 2. Suppose system (23) satisfies Assumptions 1, 2, 3 and 5. There exist an $N\times N$ matrix $Q(\tilde{\boldsymbol{x}})$, a constant $\varepsilon > 0$, and some SOS polynomials $\varepsilon_1(\boldsymbol{x}) > 0$, $\varepsilon_2(\boldsymbol{x}) > 0$, $q(\tilde{\boldsymbol{x}}) > 0$, $s(\boldsymbol{x}) \geq 0$, $s_1(\boldsymbol{x}) > 0$ such that

$$\boldsymbol{v}_1^{\mathrm{T}}(Q(\tilde{\boldsymbol{x}}) - \varepsilon I)\boldsymbol{v}_1 \quad \text{is an SOS} \tag{24}$$

$$-\boldsymbol{v}_2^{\mathrm{T}}\Xi_{opt}(\boldsymbol{x})\boldsymbol{v}_2 - \boldsymbol{v}_a^{\mathrm{T}}s(\boldsymbol{x})\Pi(\boldsymbol{x})\boldsymbol{v}_a \quad \text{is an SOS} \tag{25}$$

$$\boldsymbol{v}_3^{\mathrm{T}}\begin{bmatrix}s^2(\boldsymbol{x})(\boldsymbol{v}_a^{\mathrm{T}}\boldsymbol{v}_a)^2 & \boldsymbol{\varphi}(\tilde{\boldsymbol{x}})E_J(\boldsymbol{x})\\ * & I\end{bmatrix}\boldsymbol{v}_3 \quad \text{is an SOS} \tag{26}$$

then the state feedback stabilization problem is solvable, and a controller is given by

$$\boldsymbol{u}(\boldsymbol{x}) = -\tau^{-1}(\boldsymbol{x})\Theta(\boldsymbol{x})q(\tilde{\boldsymbol{x}})B_2^{\mathrm{T}}(\boldsymbol{x})M^{\mathrm{T}}(\boldsymbol{x})Q^{-1}(\tilde{\boldsymbol{x}})\boldsymbol{Z}(\boldsymbol{x})$$

For any initial condition, an upper bound of the output energy is given by

$$||\boldsymbol{z}||_2^2 \leq \boldsymbol{Z}^{\mathrm{T}}(\boldsymbol{x}_0)q(\tilde{\boldsymbol{x}}_0)Q^{-1}(\tilde{\boldsymbol{x}}_0)\boldsymbol{Z}(\boldsymbol{x}_0)$$

where $\Xi_{opt}(\boldsymbol{x}) = \begin{bmatrix}\Psi_q(\tilde{\boldsymbol{x}},\boldsymbol{x}) & * & *\\ \tau(\boldsymbol{x})G_1(\boldsymbol{x})Q(\tilde{\boldsymbol{x}}) & -\varepsilon_1(\boldsymbol{x})I & *\\ \tau(\boldsymbol{x})C(\boldsymbol{x})Q(\tilde{\boldsymbol{x}}) & 0 & -I\end{bmatrix}$, $\tau(\boldsymbol{x})$ and $\Theta(\boldsymbol{x})$ can be obtained from Proposition 1 by $L(\boldsymbol{x}) = G_2^{\mathrm{T}}(\boldsymbol{x})G_2(\boldsymbol{x}) + \varepsilon_2(\boldsymbol{x})D^{\mathrm{T}}(\boldsymbol{x})D(\boldsymbol{x})$, and the definitions of other parameters can be referred to Theorem 1.

Proof. Define the Lyapunov function

$$V(\boldsymbol{x}) = \boldsymbol{Z}^{\mathrm{T}}(\boldsymbol{x})P(\tilde{\boldsymbol{x}})Z(\boldsymbol{x})$$

where $P(\tilde{\boldsymbol{x}})$ is positive definite.

Let $\mathscr{F}(\boldsymbol{x},\boldsymbol{z}) = \dot{V}(\boldsymbol{x}) + \boldsymbol{z}^{\mathrm{T}}\boldsymbol{z}$. The following conclusion will be obtained by the proof of Theorem 1,

$$\mathscr{F}(\boldsymbol{x},\boldsymbol{z}) \leq 0, \text{ if}$$

$$\mathscr{F}_{OP}(p,\boldsymbol{x}) + (\boldsymbol{u} - k_1(\boldsymbol{x}))^{\mathrm{T}}\Gamma(\boldsymbol{x})(\boldsymbol{u} - k_1(\boldsymbol{x})) - k_1^{\mathrm{T}}(\boldsymbol{x})\Gamma(\boldsymbol{x})k_1(\boldsymbol{x}) \leq 0$$

where $\mathscr{F}_{OP}(p,\boldsymbol{x}) = \mathscr{F}_P(p,\boldsymbol{x}) + C^{\mathrm{T}}(\boldsymbol{x})C(\boldsymbol{x})$, $k(\boldsymbol{x}) = -\Gamma^{-1}(\boldsymbol{x})(B_2^{\mathrm{T}}(\boldsymbol{x})M^{\mathrm{T}}(\boldsymbol{x})P(\tilde{\boldsymbol{x}}) + D^{\mathrm{T}}(\boldsymbol{x})C(\boldsymbol{x}))\boldsymbol{Z}(\boldsymbol{x}), \Gamma(\boldsymbol{x}) = \varepsilon_2^{-1}(\boldsymbol{x})G_2^{\mathrm{T}}(\boldsymbol{x})G_2(\boldsymbol{x}) + D^{\mathrm{T}}(\boldsymbol{x})D(\boldsymbol{x})$.

Referring to the deduction in Theorem 1, it is easily obtained that (25) and (26) imply $\mathscr{F}(\boldsymbol{x},\boldsymbol{z}) < 0$, and an upper bound on the output energy of (23) can be determined by requiring that

$$\dot{V}(\boldsymbol{x}) < -\boldsymbol{z}^{\mathrm{T}}\boldsymbol{z}$$

Actually, this condition implies that $\dot{V}(\boldsymbol{x})<0$ and $||\boldsymbol{z}||_2^2 < V(\boldsymbol{x}_0)=\boldsymbol{Z}^{\mathrm{T}}(\boldsymbol{x}_0)P(\tilde{\boldsymbol{x}}_0)\boldsymbol{Z}(\boldsymbol{x}_0)=c$. In the end, using (24) and $\dot{V}(\boldsymbol{x})<0$, the asymptotic stability of the closed loop system is ensured. □

Theorem 2 gives an approach to determine the upper bound of the output energy (22) for system (23) by solving the SOS feasibility problem (24) ∼ (26). Furthermore, minimizing the cost of the performance objective (22) can also be obtained by solving the following SOS optimization problem

$$\min \ \mathrm{tr}(Y)$$
$$\text{s.t. } (24)\sim(26), \text{ and}$$
$$\boldsymbol{v}_4^{\mathrm{T}}\begin{bmatrix} Y & I\\ I & Q(\tilde{\boldsymbol{x}}_0)\end{bmatrix}\boldsymbol{v}_4 \quad \text{is an SOS} \tag{27}$$

where $\boldsymbol{v}_4\in\mathbf{R}^{2N}$.

Obviously, (27) is equivalent to $Q^{-1}(\tilde{\boldsymbol{x}}_0)\le Y$. Therefore, the minimal upper bound of (22) will be attained by minimizing the trace of Y, which potentially makes the upper bound of (22) closer to the true optimal value.

3.2 Nonlinear robust H_∞ control

The objective of this subsection is to design a state feedback controller under which the closed loop system is robust internally stable and has L_2-gain$\le\gamma$. The system considered in this subsection is described by (2).

Theorem 3. Suppose system (2) satisfies Assumptions 1, 2, 3 and 5. For a given scalar $\gamma>0$, there exist an $N\times N$ matrix $Q(\tilde{\boldsymbol{x}})$, a constant $\varepsilon>0$, and some SOS polynomials $\varepsilon_1(\boldsymbol{x})>0$, $\varepsilon_2(\boldsymbol{x})>0$, $q(\tilde{\boldsymbol{x}})>0$, $s(\boldsymbol{x})\ge 0$, $s_1(\boldsymbol{x})>0$ such that

$$\boldsymbol{v}_1^{\mathrm{T}}(Q(\tilde{\boldsymbol{x}})-\varepsilon I)\boldsymbol{v}_1 \quad \text{is an SOS} \tag{28}$$

$$-\boldsymbol{v}_2^{\mathrm{T}}\Xi_{H_\infty}(\boldsymbol{x})\boldsymbol{v}_2-\boldsymbol{v}_a^{\mathrm{T}}s(\boldsymbol{x})\Pi(\boldsymbol{x})\boldsymbol{v}_a \quad \text{is an SOS} \tag{29}$$

$$\boldsymbol{v}_3^{\mathrm{T}}\begin{bmatrix} s^2(\boldsymbol{x})(\boldsymbol{v}_a^{\mathrm{T}}\boldsymbol{v}_a)^2 & \boldsymbol{\varphi}(\tilde{\boldsymbol{x}})E_J(\boldsymbol{x})\\ * & I\end{bmatrix}\boldsymbol{v}_3 \quad \text{is an SOS} \tag{30}$$

then for the state feedback law

$$\boldsymbol{u}(\boldsymbol{x})=-\tau^{-1}(\boldsymbol{x})\Theta(\boldsymbol{x})q(\tilde{\boldsymbol{x}})B_2^{\mathrm{T}}(\boldsymbol{x})M^{\mathrm{T}}(\boldsymbol{x})Q^{-1}(\tilde{\boldsymbol{x}})\boldsymbol{Z}(\boldsymbol{x})$$

the closed loop system is internally stable, and has L_2-gain$\le\gamma$, where

$$\Xi_{H_\infty}(\boldsymbol{x}) = \begin{bmatrix} \Psi_q(\tilde{\boldsymbol{x}},\boldsymbol{x}) & * & * & *\\ \tau(\boldsymbol{x})G_1(\boldsymbol{x})Q(\tilde{\boldsymbol{x}}) & -\varepsilon_1(\boldsymbol{x})I & * & *\\ q(\tilde{\boldsymbol{x}})B_1^{\mathrm{T}}(\boldsymbol{x})M^{\mathrm{T}}(\boldsymbol{x}) & 0 & -\gamma^2 I & *\\ \tau(\boldsymbol{x})C(\boldsymbol{x})Q(\tilde{\boldsymbol{x}}) & 0 & 0 & -I\end{bmatrix},$$

and the definitions of other parameters can be referred to Theorem 2.

Proof. Consider the Lyapunov function candidate

$$V(\boldsymbol{x})=\boldsymbol{Z}^{\mathrm{T}}(\boldsymbol{x})P(\tilde{\boldsymbol{x}})\boldsymbol{Z}(\boldsymbol{x})$$

where $P(\tilde{\boldsymbol{x}})$ is positive definite.

Define $\mathscr{F}_{H_\infty}(\boldsymbol{x},\boldsymbol{z},\boldsymbol{w})=\dot{V}(\boldsymbol{x})+\boldsymbol{z}^{\mathrm{T}}\boldsymbol{z}-\gamma^2\boldsymbol{w}^{\mathrm{T}}\boldsymbol{w}$. It is easy to obtain the following result by the similar deduction of Theorem 2.

$$\mathscr{F}_{H_\infty}(\boldsymbol{x},\boldsymbol{z},\boldsymbol{w})<0 \text{ if (29) and (30) hold}$$

Note that $\mathscr{F}_{H_\infty}(\boldsymbol{x},\boldsymbol{z},\boldsymbol{w})<0$ implies $\dot{V}(\boldsymbol{x})<0$ with $\boldsymbol{w}\equiv\mathbf{0}$, which means the loop closed system is internally stable. On the other hand, by $\mathscr{F}_{H_\infty}(\boldsymbol{x},\boldsymbol{z},\boldsymbol{w})<0$, one has

$$V(\boldsymbol{x}(T))-V(\boldsymbol{x}(0))\le-\int_0^T(||\boldsymbol{z}(t)||^2-\gamma^2||\boldsymbol{w}(t)||^2)\mathrm{d}t$$

which implies the closed loop system has L_2-gain$\le\gamma$. □

Moreover, the optimal H_∞ control problem can also be characterized by SOS programming. In Theorem 3, let $\alpha:=\gamma^2$ as a decision variable, then the smallest γ will be solved by the following SOS optimization problem

$$\min \quad \alpha-\beta$$
$$\text{s.t. } (28)\sim(30)$$

where β is a positive number.

Remark 8. The effect of the introduction of the positive number β is the same as of the positive number ε, which is introduced to prevent numerical difficulties when solving the corresponding semi-definite program.

Theorems 1 ∼ 3 provide sufficient conditions to obtain the solutions to state feedback synthesis problem of system (2) with different performance objectives. By using SOS technique, the robust control problems have been reformulated as feasibility problems and optimization problems in SOS programming, which can be dealt with by Matlab toolbox SOSTOOLS[21].

4 Example

In this section, an example is presented to confirm the effectiveness of the approach proposed in this paper.

A nonlinear system of the form (2) is given by

$A(\boldsymbol{x})=\begin{bmatrix}-1 & 2\\ x_1 & -x_1^2-x_2^2\end{bmatrix}$, $B_1(\boldsymbol{x})=\begin{bmatrix}0\\0.1\end{bmatrix}$, $B_2(\boldsymbol{x})=\begin{bmatrix}0\\1\end{bmatrix}$

$C(\boldsymbol{x})=\begin{bmatrix}0.1 & 0\\ 0 & 0\end{bmatrix}$, $D(\boldsymbol{x})=\begin{bmatrix}0\\1\end{bmatrix}$, $\boldsymbol{Z}(\boldsymbol{x})=\begin{bmatrix}x_1\\x_2\end{bmatrix}$

The structural matrices of uncertain parameters are

$E(\boldsymbol{x})=\begin{bmatrix}0.1 & 0\\ 0 & 0.1x_1\end{bmatrix}$, $G_1(\boldsymbol{x})=\begin{bmatrix}0.1 & 0\\ 0.1 & 0\end{bmatrix}$, $G_2(\boldsymbol{x})=\begin{bmatrix}0\\0.1\end{bmatrix}$

Suppose $\Delta(\boldsymbol{x},t)=I$, $\boldsymbol{x}_0=[0.2 \quad 0.5]^{\mathrm{T}}$ and $\boldsymbol{w}=\mathbf{0}$. In Fig. 1, the open loop nonlinear system is not asymptotically stable at zero equilibrium.

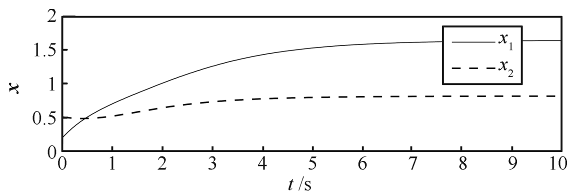

Fig. 1 State trajectories of the open loop system

From the structure of the system, the Lyapunov function can be chosen as $V(\boldsymbol{x})=\boldsymbol{Z}^{\mathrm{T}}(\boldsymbol{x})P(\tilde{x})\boldsymbol{Z}(\boldsymbol{x})$ where $\tilde{x}=x_1$.

1) State feedback stabilization

In this example, a 2-degree $Q(\tilde{x})$ is designed, and related parameters are given by $\varepsilon=0.1$, $\varepsilon_1(\boldsymbol{x})=1.1$, $\varepsilon_2(\boldsymbol{x})=0.01$, $s(\boldsymbol{x})=0.3x_1^2+0.2$, $q(\tilde{x})=x_1^2+1$, $s_1(\boldsymbol{x})=0.00001$. According to Theorem 1, the following result is obtained

$$Q(\tilde{x})=\begin{bmatrix} q_{11}(\tilde{x}) & q_{12}(\tilde{x})\\ q_{12}(\tilde{x}) & q_{22}(\tilde{x})\end{bmatrix}$$

where

$$q_{11}(\tilde{x})=0.3327x_1^2+0.1978\times10^{-8}x_1+0.3327$$
$$q_{12}(\tilde{x})=0.8\times10^{-10}x_1^2-0.46\times10^{-8}x_1+0.1\times10^{-8}$$
$$q_{22}(\tilde{x})=0.2093x_1^2-0.3712\times10^{-1}x_1+0.1782$$

And the corresponding controller is

$$\boldsymbol{u}(\boldsymbol{x}) = \frac{(8\times10^{-3}x_1^4+0.46x_1^3+0.1x_1^2+0.5x_1+0.1)1\times10^{-8}x_1}{0.0696x_1^4-0.012x_1^3+0.129x_1^2-0.012x_1+0.06} - \frac{(0.33x_1^4+2\times10^{-9}x_1^3+0.67x_1^2+2\times10^{-9}x_1+0.33)x_2}{0.0696x_1^4-0.012x_1^3+0.129x_1^2-0.012x_1+0.06}$$

Suppose $\Delta(\boldsymbol{x},t)=I$. The initial condition is chosen as $\boldsymbol{x}_0=[0.2\quad 0.5]^{\mathrm{T}}$. The state trajectories of the closed loop nonlinear system are plotted in the first subplot of Fig. 2. It is observed that the closed loop system is stable and the state trajectories converge to zero. The second subplot of Fig. 2 provides the control input profile for the state feedback controller.

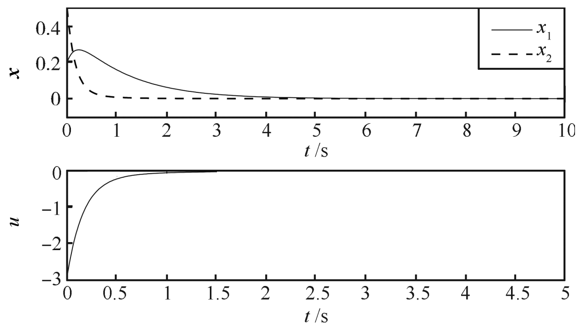

Fig. 2 State trajectories and control input

2) Nonlinear optimal control

Two algorithms of finding the upper bound of the output energy stated in Subsection 2.1 will be compared. The initial condition and system uncertainty are chosen as $\boldsymbol{x}_0=[0.2\quad 0.5]^{\mathrm{T}}$ and $\Delta(\boldsymbol{x},t)=I$, respectively. $Q(\tilde{\boldsymbol{x}})$ is again constructed as a 2-degree polynomial matrix. Other related parameters are given by $\varepsilon=0.1$, $\varepsilon_1(\boldsymbol{x})=1.1$, $\varepsilon_2(\boldsymbol{x})=0.1$, $s(\boldsymbol{x})=0.3x_1^2+0.2$, $q(\tilde{\boldsymbol{x}})=x_1^2+1$, $s_1(\boldsymbol{x})=0.00001$. The computation results are listed in Table 1, where Algorithm 1 denotes the method of Theorem 2 and Algorithm 2 denotes the method of getting the optimal solution of Theorem 2 based on optimization theory in SOS programming. As expected, the upper bound of output energy decreases after minimizing the $\mathrm{tr}(Y)$. Besides, the upper bound $c=x_0^{\mathrm{T}}q(\tilde{x}_0)Q^{-1}(\tilde{x}_0)\boldsymbol{x}_0$ is closer to the actual value $\|\boldsymbol{z}\|_2^2$ by Algorithm 2. The results show the conclusion stated in Subsection 2.1.

Table 1 Results of the controller design in Subsection 3.1

	Algorithm 1	Algorithm 2
$\mathrm{tr}(Y)$		3.5008
$c=\boldsymbol{x}_0^{\mathrm{T}}q(\tilde{x}_0)Q^{-1}(\tilde{x}_0)\boldsymbol{x}_0$	1.7346	0.672
Actual value $\|\boldsymbol{z}\|_2^2$	0.6804	0.2962

3) Nonlinear robust H_∞ control

Suppose the disturbance $\boldsymbol{w}(t)$ acting on the system is

$$\boldsymbol{w}(t)=\begin{cases}0.05(\sin(0.05t)+\cos(0.05t)), & 0<t\le 10\\ 0, & t>10\end{cases}$$

Then, the optimal disturbance attenuation level can be obtained by optimization algorithm of Subsection 2.2. In addition, a state feedback robust H_∞ controller is also designed using the result stated in Theorem 3. Now, $Q(\tilde{x})$ is designed as a 2-degree polynomial matrix, and other parameters are chosen as $\varepsilon=0.1$, $\varepsilon_1(\boldsymbol{x})=1.1$, $\varepsilon_2(\boldsymbol{x})=0.1$, $s(\boldsymbol{x})=0.3x_1^2+0.2$, $q(\tilde{x})=x_1^2+1$, $s_1(\boldsymbol{x})=0.00001$. The optimization algorithm, with $\beta=0.1$, returns $\gamma_{opt}=0.14$, which means the L_2-gain from $\boldsymbol{w}$ to $\boldsymbol{z}$ of the closed loop system is no greater than 0.14. The truncated norms $\|\boldsymbol{z}\|_{2,t}/\|\boldsymbol{w}\|_{2,t}$, $t\in[0,T]$ with zero initial condition and $\Delta(\boldsymbol{x},t)=I$ are depicted in Fig. 3. As can be seen, the truncated norms are indeed less than 0.14.

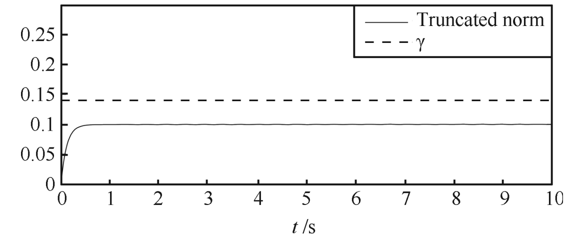

Fig. 3 Closed loop optimal disturbance attenuation index

Furthermore, the L_2-gain of the closed loop system is

$$\frac{\int_0^T\|\boldsymbol{z}\|^2\mathrm{d}t}{\int_0^T\|\boldsymbol{w}\|^2\mathrm{d}t}=0.98\times10^{-2}\le\gamma_{opt}^2=0.02,\quad T=10$$

Obviously, the robust H_∞ control problem of Theorem 3 is solvable, if $\gamma\ge0.14$. Now, γ is fixed at 0.5 in Theorem 3, then the result returns

$$Q(\tilde{x})=\begin{bmatrix}q_{11}(\tilde{x}) & q_{12}(\tilde{x})\\ q_{12}(\tilde{x}) & q_{22}(\tilde{x})\end{bmatrix}$$

where

$$q_{11}(\tilde{x})=0.3046x_1^2+0.3\times10^{-8}x_1+0.3046$$
$$q_{12}(\tilde{x})=-0.41\times10^{-10}x_1^2-0.85\times10^{-8}x_1+0.33\times10^{-8}$$
$$q_{22}(\tilde{x})=0.1816x_1^2-0.3722\times10^{-1}x_1+0.16097$$

And the following controller can be obtained

$$\boldsymbol{u}(\boldsymbol{x}) = -\frac{(0.28x_1^4+3\times10^{-9}x_1^3+0.6x_1^2+3\times10^{-9}x_1+0.28)x_2}{0.055x_1^4-0.011x_1^3+0.105x_1^2-0.011x_1+0.049} - \frac{(0.004x_1^4+0.8x_1^3-0.3x_1^2+0.8x_1-0.3)\times10^{-8}x_1}{0.055x_1^4-0.011x_1^3+0.105x_1^2-0.011x_1+0.049}$$

What are plotted in Fig. 4 include the state trajectories and control input of the closed loop system under state feedback law $\boldsymbol{u}(\boldsymbol{x})$, with initial condition $\boldsymbol{x}_0=[0.2\quad 0.5]^{\mathrm{T}}$ and $\Delta(\boldsymbol{x},t)=I$.

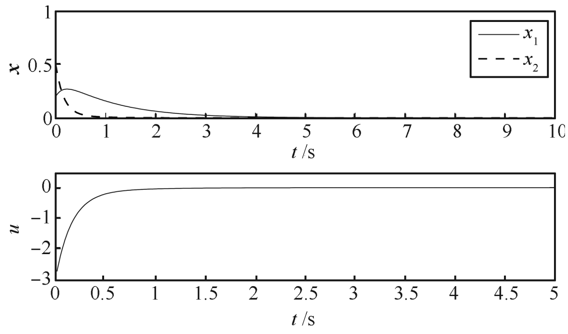

Fig. 4 State trajectories and control input

5 Conclusion

This paper addressed the robust state feedback synthesis problem for a class of uncertain polynomial nonlinear systems without and with guaranteed cost and H_∞ performance objectives. By using sum of squares technique, the above control problems were formulated as some convex sum of squares programming problems. Furthermore, the corresponding solvability conditions for robust stabilization without and with (optimal) guaranteed cost and H_∞ performance objectives were given. It should be pointed out that all these solvability conditions can be solved by a third-party Matlab toolbox SOSTOOLS, which means that the difficulty of calculation in nonlinear systems will be raveled out to some extent. As the approach proposed in the paper only applies to a class of polynomial systems, there are still many issues worthy of study. For example, how to extend the approach to more complex nonlinear systems, which is an interesting problem, needs to be studied further.

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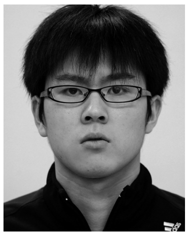

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